# 6

# *Markov Chains*

**Contents**

The Bernoulli and Poisson processes studied in the preceding chapter are memoryless, in the sense that the future does not depend on the past: the occurrences of new "successes" or "arrivals" do not depend on the past history of the process. In this chapter, we consider processes where the future depends on and can be predicted to some extent by what has happened in the past.

We emphasize models where the effect of the past on the future is summarized by a **state**, which changes over time according to given probabilities. We restrict ourselves to models whose state can take a finite number of values and can change in discrete instants of time. We want to analyze the probabilistic properties of the sequence of state values.

The range of applications of the models of this chapter is truly vast. It includes just about any dynamical system whose evolution over time involves uncertainty, provided the state of the system is suitably defined. Such systems arise in a broad variety of fields, such as communications, automatic control, signal processing, manufacturing, economics, resource allocation, etc.

## 6.1 DISCRETE-TIME MARKOV CHAINS

We will first consider **discrete-time Markov chains**, in which the state changes at certain discrete time instants, indexed by an integer variable $n$. At each time step $n$, the Markov chain has a **state**, denoted by $X_n$, which belongs to a **finite** set $\mathcal{S}$ of possible states, called the **state space**. Without loss of generality, and unless there is a statement to the contrary, we will assume that $\mathcal{S} = \{1, \ldots, m\}$, for some positive integer $m$. The Markov chain is described in terms of its **transition probabilities** $p_{ij}$: whenever the state happens to be $i$, there is probability $p_{ij}$ that the next state is equal to $j$. Mathematically,

$$p_{ij} = \mathbf{P}(X_{n+1} = j \mid X_n = i), \qquad i,\, j \in \mathcal{S}.$$

The key assumption underlying Markov processes is that the transition probabilities $p_{ij}$ apply whenever state $i$ is visited, no matter what happened in the past, and no matter how state $i$ was reached. Mathematically, we assume the **Markov property**, which requires that

$$\begin{aligned}\mathbf{P}(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) &= \mathbf{P}(X_{n+1} = j \mid X_n = i) \\ &= p_{ij},\end{aligned}$$

for all times $n$, all states $i, j \in \mathcal{S}$, and all possible sequences $i_0, \ldots, i_{n-1}$ of earlier states. Thus, the probability law of the next state $X_{n+1}$ depends on the past only through the value of the present state $X_n$.

The transition probabilities $p_{ij}$ must be of course nonnegative, and sum to one:

$$\sum_{j=1}^{m} p_{ij} = 1, \qquad \text{for all } i.$$

We will generally allow the probabilities $p_{ii}$ to be positive, in which case it is possible for the next state to be the same as the current one. Even though the state does not change, we still view this as a state transition of a special type (a "self-transition").

**Specification of Markov Models**

- A Markov chain model is specified by identifying
  (a) the set of states $\mathcal{S} = \{1, \ldots, m\}$,
  (b) the set of possible transitions, namely, those pairs $(i, j)$ for which $p_{ij} > 0$, and,
  (c) the numerical values of those $p_{ij}$ that are positive.
- The Markov chain specified by this model is a sequence of random variables $X_0, X_1, X_2, \ldots$, that take values in $\mathcal{S}$ and which satisfy

$$\mathbf{P}(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = p_{ij},$$

  for all times $n$, all states $i, j \in \mathcal{S}$, and all possible sequences $i_0, \ldots, i_{n-1}$ of earlier states.

All of the elements of a Markov chain model can be encoded in a **transition probability matrix**, which is simply a two-dimensional array whose element at the $i$th row and $j$th column is $p_{ij}$:

$$\begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix}.$$

It is also helpful to lay out the model in the so-called **transition probability graph**, whose nodes are the states and whose arcs are the possible transitions. By recording the numerical values of $p_{ij}$ near the corresponding arcs, one can visualize the entire model in a way that can make some of its major properties readily apparent.

**Example 6.1.** Alice is taking a probability class and in each week she can be either up-to-date or she may have fallen behind. If she is up-to-date in a given week, the probability that she will be up-to-date (or behind) in the next week is 0.8 (or 0.2, respectively). If she is behind in the given week, the probability that she will be up-to-date (or behind) in the next week is 0.6 (or 0.4, respectively). We assume that these probabilities do not depend on whether she was up-to-date or behind in previous weeks, so the problem has the typical Markov chain character (the future depends on the past only through the present).

Let us introduce states 1 and 2, and identify them with being up-to-date and behind, respectively. Then, the transition probabilities are

$$p_{11} = 0.8, \qquad p_{12} = 0.2, \qquad p_{21} = 0.6, \qquad p_{22} = 0.4,$$

and the transition probability matrix is

$$\begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix}.$$

The transition probability graph is shown in Fig. 6.1.

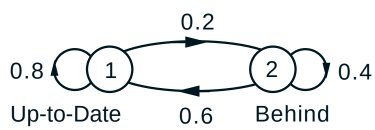



**Figure 6.1:** The transition probability graph in Example 6.1.

**Example 6.2.** A fly moves along a straight line in unit increments. At each time period, it moves one unit to the left with probability 0.3, one unit to the right with probability 0.3, and stays in place with probability 0.4, independently of the past history of movements. A spider is lurking at positions 1 and $m$: if the fly lands there, it is captured by the spider, and the process terminates. We want to construct a Markov chain model, assuming that the fly starts in one of the positions $2, \ldots, m-1$.

Let us introduce states $1, 2, \ldots, m$, and identify them with the corresponding positions of the fly. The nonzero transition probabilities are

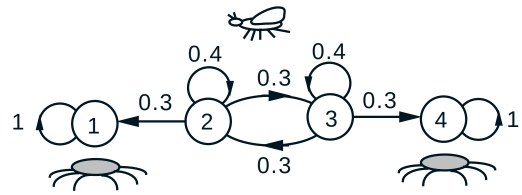
$$p_{11} = 1, \qquad p_{mm} = 1,$$

$$p_{ij} = \begin{cases} 0.3 & \text{if } j = i-1 \text{ or } j = i+1, \\ 0.4 & \text{if } j = i, \end{cases} \qquad \text{for } i = 2, \ldots, m-1.$$

The transition probability graph and matrix are shown in Fig. 6.2.

Given a Markov chain model, we can compute the probability of any particular sequence of future states. This is analogous to the use of the multiplication rule in sequential (tree) probability models. In particular, we have

$$\mathbf{P}(X_0 = i_0, X_1 = i_1, \ldots, X_{i_n} = i_n) = \mathbf{P}(X_0 = i_0) p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{n-1} i_n}.$$




| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1.0 | 0 | 0 | 0 |
| 2 | 0.3 | 0.4 | 0.3 | 0 |
| 3 | 0 | 0.3 | 0.4 | 0.3 |
| 4 | 0 | 0 | 0 | 1.0 |

$p_{ij}$

**Figure 6.2:** The transition probability graph and the transition probability matrix in Example 6.2, for the case where $m = 4$.

To verify this property, note that

$$\begin{aligned}
&\mathbf{P}(X_0 = i_0, X_1 = i_1, \ldots, X_{i_n} = i_n) \\
&\qquad = \mathbf{P}(X_n = i_n \,|\, X_0 = i_0, \ldots, X_{n-1} = i_{n-1})\mathbf{P}(X_0 = i_0, \ldots, X_{n-1} = i_{n-1}) \\
&\qquad = p_{i_{n-1}i_n}\mathbf{P}(X_0 = i_0, \ldots, X_{n-1} = i_{n-1}),
\end{aligned}$$

where the last equality made use of the Markov property. We then apply the same argument to the term $\mathbf{P}(X_0 = i_0, \ldots, X_{n-1} = i_{n-1})$ and continue similarly, until we eventually obtain the desired expression. If the initial state $X_0$ is given and is known to be equal to some $i_0$, a similar argument yields

$$\mathbf{P}(X_1 = i_1, \ldots, X_{i_n} = i_n \,|\, X_0 = i_0) = p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{n-1} i_n}.$$

Graphically, a state sequence can be identified with a sequence of arcs in the transition probability graph, and the probability of such a path (given the initial state) is given by the product of the probabilities associated with the arcs traversed by the path.

**Example 6.3.** For the spider and fly example (Example 6.2), we have

$$\mathbf{P}(X_1 = 2, X_2 = 2, X_3 = 3, X_4 = 4 \,|\, X_0 = 2) = p_{22}p_{22}p_{23}p_{34} = (0.4)^2(0.3)^2.$$

We also have

$$\begin{aligned}
\mathbf{P}(X_0 = 2, X_1 = 2, X_2 = 2, X_3 = 3, X_4 = 4) &= \mathbf{P}(X_0 = 2)p_{22}p_{22}p_{23}p_{34} \\
&= \mathbf{P}(X_0 = 2)(0.4)^2(0.3)^2.
\end{aligned}$$

Note that in order to calculate a probability of this form, in which there is no conditioning on a fixed initial state, we need to specify a probability law for the initial state $X_0$.

### $n$-Step Transition Probabilities

Many Markov chain problems require the calculation of the probability law of the state at some future time, conditioned on the current state. This probability law is captured by the **$n$-step transition probabilities**, defined by

$$r_{ij}(n) = \mathbf{P}(X_n = j \mid X_0 = i).$$

In words, $r_{ij}(n)$ is the probability that the state after $n$ time periods will be $j$, given that the current state is $i$. It can be calculated using the following basic recursion, known as the **Chapman-Kolmogorov equation**.

**Chapman-Kolmogorov Equation for the $n$-Step Transition Probabilities**

The $n$-step transition probabilities can be generated by the recursive formula

$$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1)p_{kj}, \qquad \text{for } n > 1, \quad \text{and all } i,\ j,$$

starting with

$$r_{ij}(1) = p_{ij}.$$

To verify the formula, we apply the total probability theorem as follows:

$$\begin{aligned}\mathbf{P}(X_n = j \mid X_0 = i) &= \sum_{k=1}^{m} \mathbf{P}(X_{n-1} = k \mid X_0 = i) \cdot \mathbf{P}(X_n = j \mid X_{n-1} = k,\ X_0 = i) \\ &= \sum_{k=1}^{m} r_{ik}(n-1)p_{kj};\end{aligned}$$

see Fig. 6.3 for an illustration. We have used here the Markov property: once we condition on $X_{n-1} = k$, the conditioning on $X_0 = i$ does not affect the probability $p_{kj}$ of reaching $j$ at the next step.

We can view $r_{ij}(n)$ as the element at the $i$th row and $j$th column of a two-dimensional array, called the **$n$-step transition probability matrix**.† Figures

† Those readers familiar with matrix multiplication, may recognize that the Chapman-Kolmogorov equation can be expressed as follows: the matrix of $n$-step transition probabilities $r_{ij}(n)$ is obtained by multiplying the matrix of $(n-1)$-step transition probabilities $r_{ik}(n-1)$, with the one-step transition probability matrix. Thus, the $n$-step transition probability matrix is the $n$th power of the transition probability matrix.

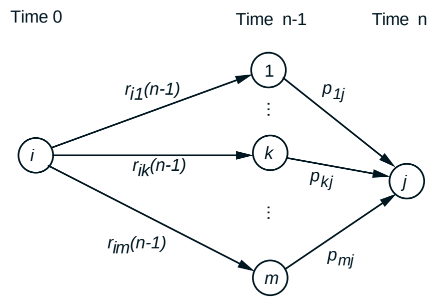
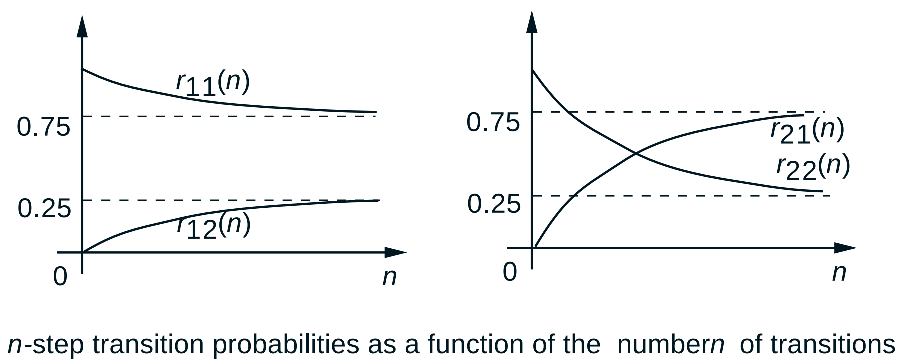



**Figure 6.3:** Derivation of the Chapman-Kolmogorov equation. The probability of being at state $j$ at time $n$ is the sum of the probabilities $r_{ik}(n-1)p_{kj}$ of the different ways of reaching $j$.




| | UpD | B |
|---|---|---|
| UpD | 0.8 | 0.2 |
| B | 0.6 | 0.4 |

$r_{ij}(1)$

| | |
|---|---|
| .76 | .24 |
| .72 | .28 |

$r_{ij}(2)$

| | |
|---|---|
| .752 | .248 |
| .744 | .256 |

$r_{ij}(3)$

| | |
|---|---|
| .7504 | .2496 |
| .7488 | .2512 |

$r_{ij}(4)$

| | |
|---|---|
| .7501 | .2499 |
| .7498 | .2502 |

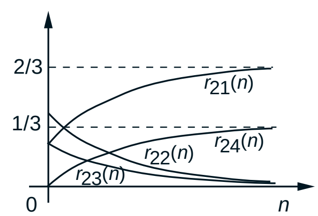
$r_{ij}(5)$

Sequence of *n* -step transition probability matrices

**Figure 6.4:** $n$-step transition probabilities for the "up-to-date/behind" Example 6.1. Observe that as $n \to \infty$, $r_{ij}(n)$ converges to a limit that does not depend on the initial state.

6.4 and 6.5 give the $n$-step transition probabilities $r_{ij}(n)$ for the cases of Examples 6.1 and 6.2, respectively. There are some interesting observations about the limiting behavior of $r_{ij}(n)$ in these two examples. In Fig. 6.4, we see that each $r_{ij}(n)$ converges to a limit, as $n \to \infty$, and this limit does not depend on the initial state. Thus, each state has a positive "steady-state" probability of being occupied at times far into the future. Furthermore, the probability $r_{ij}(n)$ depends on the initial state $i$ when $n$ is small, but over time this dependence diminishes. Many (but by no means all) probabilistic models that evolve over time have such a character: after a sufficiently long time, the effect of their initial condition becomes negligible.

In Fig. 6.5, we see a qualitatively different behavior: $r_{ij}(n)$ again converges, but the limit depends on the initial state, and can be zero for selected states. Here, we have two states that are "absorbing," in the sense that they are infinitely repeated, once reached. These are the states 1 and 4 that correspond to the capture of the fly by one of the two spiders. Given enough time, it is certain that some absorbing state will be reached. Accordingly, the probability of being at the non-absorbing states 2 and 3 diminishes to zero as time increases.




*n*-step transition probabilities as a function of the time *n*

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1.0 | 0 | 0 | 0 |
| 2 | 0.3 | 0.4 | 0.3 | 0 |
| 3 | 0 | 0.3 | 0.4 | 0.3 |
| 4 | 0 | 0 | 0 | 1.0 |

$r_{ij}(1)$

| | | | |
|---|---|---|---|
| 1.0 | 0 | 0 | 0 |
| .42 | .25 | .24 | .09 |
| .09 | .24 | .25 | .42 |
| 0 | 0 | 0 | 1.0 |

$r_{ij}(2)$

| | | | |
|---|---|---|---|
| 1.0 | 0 | 0 | 0 |
| .50 | .17 | .17 | .16 |
| .16 | .17 | .17 | .50 |
| 0 | 0 | 0 | 1.0 |

$r_{ij}(3)$

| | | | |
|---|---|---|---|
| 1.0 | 0 | 0 | 0 |
| .55 | .12 | .12 | .21 |
| .21 | .12 | .12 | .55 |
| 0 | 0 | 0 | 1.0 |

$r_{ij}(4)$

....

| | | | |
|---|---|---|---|
| 1.0 | 0 | 0 | 0 |
| 2/3 | 0 | 0 | 1/3 |
| 1/3 | 0 | 0 | 2/3 |
| 0 | 0 | 0 | 1.0 |

$r_{ij}(\infty)$

Sequence of transition probability matrices

**Figure 6.5:** $n$-step transition probabilities for the "spiders-and-fly" Example 6.2. Observe that $r_{ij}(n)$ converges to a limit that depends on the initial state.

These examples illustrate that there is a variety of types of states and asymptotic occupancy behavior in Markov chains. We are thus motivated to classify and analyze the various possibilities, and this is the subject of the next three sections.

## 6.2 CLASSIFICATION OF STATES

In the preceding section, we saw through examples several types of Markov chain states with qualitatively different characteristics. In particular, some states, after being visited once, are certain to be revisited again, while for some other states this may not be the case. In this section, we focus on the mechanism by which this occurs. In particular, we wish to classify the states of a Markov chain with a focus on the long-term frequency with which they are visited.

As a first step, we make the notion of revisiting a state precise. Let us say that a state $j$ is **accessible** from a state $i$ if for some $n$, the $n$-step transition probability $r_{ij}(n)$ is positive, i.e., if there is positive probability of reaching $j$, starting from $i$, after some number of time periods. An equivalent definition is that there is a possible state sequence $i, i_1, \ldots, i_{n-1}, j$, that starts at $i$ and ends at $j$, in which the transitions $(i, i_1), (i_1, i_2), \ldots, (i_{n-2}, i_{n-1}), (i_{n-1}, j)$ all have positive probability. Let $A(i)$ be the set of states that are accessible from $i$. We say that $i$ is **recurrent** if for every $j$ that is accessible from $i$, $i$ is also accessible from $j$; that is, for all $j$ that belong to $A(i)$ we have that $i$ belongs to $A(j)$.

When we start at a recurrent state $i$, we can only visit states $j \in A(i)$ from which $i$ is accessible. Thus, from any future state, there is always some probability of returning to $i$ and, given enough time, this is certain to happen. By repeating this argument, if a recurrent state is visited once, it will be revisited an infinite number of times.

A state is called **transient** if it is not recurrent. In particular, there are states $j \in A(i)$ such that $i$ is not accessible from $j$. After each visit to state $i$, there is positive probability that the state enters such a $j$. Given enough time, this will happen, and state $i$ cannot be visited after that. Thus, a transient state will only be visited a finite number of times.

Note that transience or recurrence is determined by the arcs of the transition probability graph [those pairs $(i, j)$ for which $p_{ij} > 0$] and not by the numerical values of the $p_{ij}$. Figure 6.6 provides an example of a transition probability graph, and the corresponding recurrent and transient states.

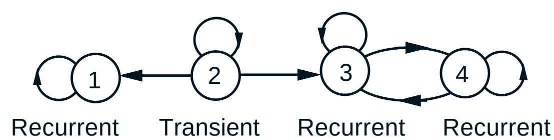



**Figure 6.6:** Classification of states given the transition probability graph. Starting from state 1, the only accessible state is itself, and so 1 is a recurrent state. States 1, 3, and 4 are accessible from 2, but 2 is not accessible from any of them, so state 2 is transient. States 3 and 4 are accessible only from each other (and themselves), and they are both recurrent.

If $i$ is a recurrent state, the set of states $A(i)$ that are accessible from $i$

form a **recurrent class** (or simply **class**), meaning that states in $A(i)$ are all accessible from each other, and no state outside $A(i)$ is accessible from them. Mathematically, for a recurrent state $i$, we have $A(i) = A(j)$ for all $j$ that belong to $A(i)$, as can be seen from the definition of recurrence. For example, in the graph of Fig. 6.6, states 3 and 4 form a class, and state 1 by itself also forms a class.

It can be seen that at least one recurrent state must be accessible from any given transient state. This is intuitively evident, and a more precise justification is given in the theoretical problems section. It follows that there must exist at least one recurrent state, and hence at least one class. Thus, we reach the following conclusion.

**Markov Chain Decomposition**

- A Markov chain can be decomposed into one or more recurrent classes, plus possibly some transient states.
- A recurrent state is accessible from all states in its class, but is not accessible from recurrent states in other classes.
- A transient state is not accessible from any recurrent state.
- At least one, possibly more, recurrent states are accessible from a given transient state.

Figure 6.7 provides examples of Markov chain decompositions. Decomposition provides a powerful conceptual tool for reasoning about Markov chains and visualizing the evolution of their state. In particular, we see that:

(a) once the state enters (or starts in) a class of recurrent states, it stays within that class; since all states in the class are accessible from each other, all states in the class will be visited an infinite number of times;

(b) if the initial state is transient, then the state trajectory contains an initial portion consisting of transient states and a final portion consisting of recurrent states from the same class.

For the purpose of understanding long-term behavior of Markov chains, it is important to analyze chains that consist of a single recurrent class. For the purpose of understanding short-term behavior, it is also important to analyze the mechanism by which any particular class of recurrent states is entered starting from a given transient state. These two issues, long-term and short-term behavior, are the focus of Sections 6.3 and 6.4, respectively.

### Periodicity

One more characterization of a recurrent class is of special interest, and relates

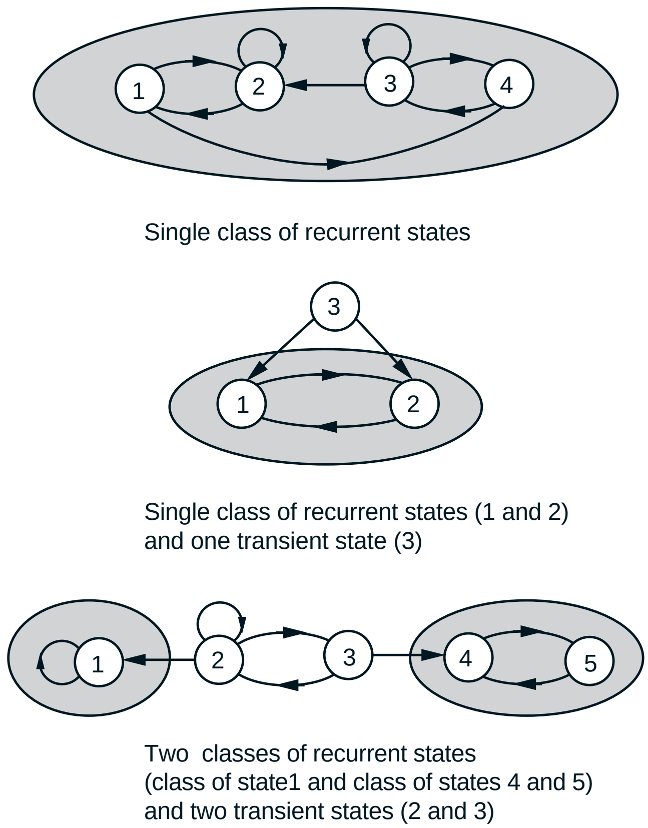



**Figure 6.7:** Examples of Markov chain decompositions into recurrent classes and transient states.

to the presence or absence of a certain periodic pattern in the times that a state is visited. In particular, a recurrent class is said to be **periodic** if its states can be grouped in $d > 1$ disjoint subsets $S_1, \ldots, S_d$ so that all transitions from one subset lead to the next subset; see Fig. 6.8. More precisely,

$$\text{if} \quad i \in S_k \text{ and } p_{ij} > 0, \quad \text{then } \begin{cases} j \in S_{k+1}, & \text{if } k = 1, \ldots, d-1, \\ j \in S_1, & \text{if } k = d. \end{cases}$$

A recurrent class that is not periodic, is said to be **aperiodic**.

Thus, in a periodic recurrent class, we move through the sequence of subsets in order, and after $d$ steps, we end up in the same subset. As an example, the recurrent class in the second chain of Fig. 6.7 (states 1 and 2) is periodic, and the same is true of the class consisting of states 4 and 5 in the third chain of Fig. 6.7. All other classes in the chains of this figure are aperiodic.

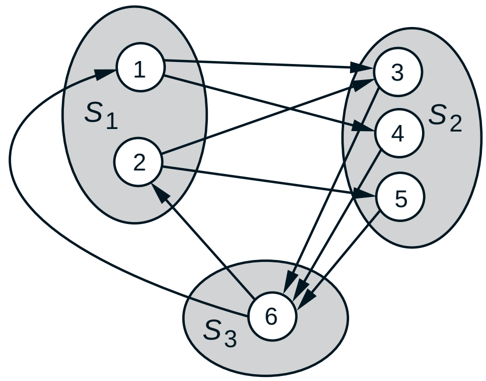



**Figure 6.8:** Structure of a periodic recurrent class.

Note that given a periodic recurrent class, a positive time $n$, and a state $j$ in the class, there must exist some state $i$ such that $r_{ij}(n) = 0$. The reason is that, from the definition of periodicity, the states are grouped in subsets $S_1, \ldots, S_d$, and the subset to which $j$ belongs can be reached at time $n$ from the states in only one of the subsets. Thus, a way to verify aperiodicity of a given recurrent class $R$, is to check whether there is a special time $\overline{n} \geq 1$ and a special state $s \in R$ that can be reached at time $\overline{n}$ from all initial states in $R$, i.e., $r_{is}(\overline{n}) > 0$ for all $i \in R$. As an example, consider the first chain in Fig. 6.7. State $s = 2$ can be reached at time $\overline{n} = 2$ starting from every state, so the unique recurrent class of that chain is aperiodic.

A converse statement, which we do not prove, also turns out to be true: if a recurrent class is not periodic, then a time $\overline{n}$ and a special state $s$ with the above properties can always be found.

**Periodicity**

Consider a recurrent class $R$.

- The class is called **periodic** if its states can be grouped in $d > 1$ disjoint subsets $S_1, \ldots, S_d$, so that all transitions from $S_k$ lead to $S_{k+1}$ (or to $S_1$ if $k = d$).
- The class is **aperiodic** (not periodic) if and only if there exists a time $\overline{n}$ and a state $s$ in the class, such that $p_{is}(\overline{n}) > 0$ for all $i \in R$.

## 6.3 STEADY-STATE BEHAVIOR

In Markov chain models, we are often interested in long-term state occupancy behavior, that is, in the $n$-step transition probabilities $r_{ij}(n)$ when $n$ is very large. We have seen in the example of Fig. 6.4 that the $r_{ij}(n)$ may converge to steady-state values that are independent of the initial state, so to what extent is this behavior typical?

If there are two or more classes of recurrent states, it is clear that the limiting values of the $r_{ij}(n)$ must depend on the initial state (visiting $j$ far into the future will depend on whether $j$ is in the same class as the initial state $i$). We will, therefore, restrict attention to chains involving a single recurrent class, plus possibly some transient states. This is not as restrictive as it may seem, since we know that once the state enters a particular recurrent class, it will stay within that class. Thus, asymptotically, the presence of all classes except for one is immaterial.

Even for chains with a single recurrent class, the $r_{ij}(n)$ may fail to converge. To see this, consider a recurrent class with two states, 1 and 2, such that from state 1 we can only go to 2, and from 2 we can only go to 1 ($p_{12} = p_{21} = 1$). Then, starting at some state, we will be in that same state after any even number of transitions, and in the other state after any odd number of transitions. What is happening here is that the recurrent class is periodic, and for such a class, it can be seen that the $r_{ij}(n)$ generically oscillate.

We now assert that for every state $j$, the $n$-step transition probabilities $r_{ij}(n)$ approach a limiting value that is independent of $i$, provided we exclude the two situations discussed above (multiple recurrent classes and/or a periodic class). This limiting value, denoted by $\pi_j$, has the interpretation

$$\pi_j \approx \mathbf{P}(X_n = j), \qquad \text{when } n \text{ is large,}$$

and is called the **steady-state probability of** $j$. The following is an important theorem. Its proof is quite complicated and is outlined together with several other proofs in the theoretical problems section.

**Steady-State Convergence Theorem**

Consider a Markov chain with a single recurrent class, which is aperiodic. Then, the states $j$ are associated with steady-state probabilities $\pi_j$ that have the following properties.

(a) $$\lim_{n\to\infty} r_{ij}(n) = \pi_j, \qquad \text{for all } i,\ j.$$

(b) The $\pi_j$ are the unique solution of the system of equations below:

$$\pi_j = \sum_{k=1}^{m} \pi_k p_{kj}, \qquad j = 1, \ldots, m,$$
$$1 = \sum_{k=1}^{m} \pi_k.$$

(c) We have

$$\pi_j = 0, \qquad \text{for all transient states } j,$$
$$\pi_j > 0, \qquad \text{for all recurrent states } j.$$

Since the steady-state probabilities $\pi_j$ sum to 1, they form a probability distribution on the state space, called the **stationary distribution** of the chain. The reason for the name is that if the initial state is chosen according to this distribution, i.e., if

$$\mathbf{P}(X_0 = j) = \pi_j, \qquad j = 1, \ldots, m,$$

then, using the total probability theorem, we have

$$\mathbf{P}(X_1 = j) = \sum_{k=1}^{m} \mathbf{P}(X_0 = k) p_{kj} = \sum_{k=1}^{m} \pi_k p_{kj} = \pi_j,$$

where the last equality follows from part (b) of the steady-state convergence theorem. Similarly, we obtain $\mathbf{P}(X_n = j) = \pi_j$, for all $n$ and $j$. Thus, if the initial state is chosen according to the stationary distribution, all subsequent states will have the same distribution.

The equations

$$\pi_j = \sum_{k=1}^{m} \pi_k p_{kj}, \qquad j = 1, \ldots, m,$$

are called the **balance equations**. They are a simple consequence of part (a) of the theorem and the Chapman-Kolmogorov equation. Indeed, once the convergence of $r_{ij}(n)$ to some $\pi_j$ is taken for granted, we can consider the equation,

$$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1) p_{kj},$$

take the limit of both sides as $n \to \infty$, and recover the balance equations.† The balance equations are a linear system of equations that, together with $\sum_{k=1}^{m} \pi_k = 1$, can be solved to obtain the $\pi_j$. The following examples illustrate the solution process.

**Example 6.4.** Consider a two-state Markov chain with transition probabilities

$$p_{11} = 0.8, \qquad p_{12} = 0.2,$$

$$p_{21} = 0.6, \qquad p_{22} = 0.4.$$

[This is the same as the chain of Example 6.1 (cf. Fig. 6.1).] The balance equations take the form

$$\pi_1 = \pi_1 p_{11} + \pi_2 p_{21}, \qquad \pi_2 = \pi_1 p_{12} + \pi_2 p_{22},$$

or

$$\pi_1 = 0.8 \cdot \pi_1 + 0.6 \cdot \pi_2, \qquad \pi_2 = 0.2 \cdot \pi_1 + 0.4 \cdot \pi_2.$$

Note that the above two equations are dependent, since they are both equivalent to

$$\pi_1 = 3\pi_2.$$

This is a generic property, and in fact it can be shown that one of the balance equations depends on the remaining equations (see the theoretical problems). However, we know that the $\pi_j$ satisfy the normalization equation

$$\pi_1 + \pi_2 = 1,$$

which supplements the balance equations and suffices to determine the $\pi_j$ uniquely. Indeed, by substituting the equation $\pi_1 = 3\pi_2$ into the equation $\pi_1 + \pi_2 = 1$, we obtain $3\pi_2 + \pi_2 = 1$, or

$$\pi_2 = 0.25,$$

which using the equation $\pi_1 + \pi_2 = 1$, yields

$$\pi_1 = 0.75.$$

This is consistent with what we found earlier by iterating the Chapman-Kolmogorov equation (cf. Fig. 6.4).

**Example 6.5.** An absent-minded professor has two umbrellas that she uses when commuting from home to office and back. If it rains and an umbrella is available in

† According to a famous and important theorem from linear algebra (called the Perron-Frobenius theorem), the balance equations always have a nonnegative solution, for any Markov chain. What is special about a chain that has a single recurrent class, which is aperiodic, is that the solution is unique and is also equal to the limit of the $n$-step transition probabilities $r_{ij}(n)$.

her location, she takes it. If it is not raining, she always forgets to take an umbrella. Suppose that it rains with probability $p$ each time she commutes, independently of other times. What is the steady-state probability that she gets wet on a given day?

We model this problem using a Markov chain with the following states:

State $i$: $i$ umbrellas are available in her current location, $\quad i = 0, 1, 2.$

The transition probability graph is given in Fig. 6.9, and the transition probability matrix is

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{bmatrix}.$$

The chain has a single recurrent class that is aperiodic (assuming $0 < p < 1$), so the steady-state convergence theorem applies. The balance equations are

$$\pi_0 = (1-p)\pi_2, \quad \pi_1 = (1-p)\pi_1 + p\pi_2, \quad \pi_2 = \pi_0 + p\pi_1.$$

From the second equation, we obtain $\pi_1 = \pi_2$, which together with the first equation $\pi_0 = (1-p)\pi_2$ and the normalization equation $\pi_0 + \pi_1 + \pi_2 = 1$, yields

$$\pi_0 = \frac{1-p}{3-p}, \quad \pi_1 = \frac{1}{3-p}, \quad \pi_2 = \frac{1}{3-p}.$$

According to the steady-state convergence theorem, the steady-state probability that the professor finds herself in a place without an umbrella is $\pi_0$. The steady-state probability that she gets wet is $\pi_0$ times the probability of rain $p$.

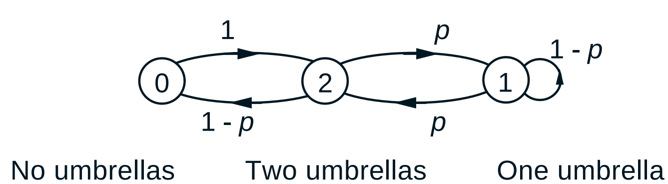



**Figure 6.9:** Transition probability graph for Example 6.5.

**Example 6.6.** A superstitious professor works in a circular building with $m$ doors, where $m$ is odd, and never uses the same door twice in a row. Instead he uses with probability $p$ (or probability $1-p$) the door that is adjacent in the clockwise direction (or the counterclockwise direction, respectively) to the door he used last. What is the probability that a given door will be used on some particular day far into the future?

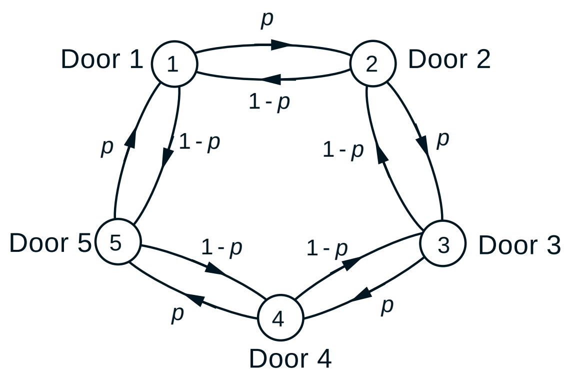



**Figure 6.10:** Transition probability graph in Example 6.6, for the case of $m = 5$ doors.

We introduce a Markov chain with the following $m$ states:

$$\text{State } i\text{: Last door used is door } i, \qquad i = 1, \ldots, m.$$

The transition probability graph of the chain is given in Fig. 6.10, for the case $m = 5$. The transition probability matrix is

$$\begin{bmatrix} 0 & p & 0 & 0 & \ldots & 0 & 1-p \\ 1-p & 0 & p & 0 & \ldots & 0 & 0 \\ 0 & 1-p & 0 & p & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p & 0 & 0 & 0 & \ldots & 1-p & 0 \end{bmatrix}.$$

Assuming that $0 < p < 1$, the chain has a single recurrent class that is aperiodic. [To verify aperiodicity, argue by contradiction: if the class were periodic, there could be only two subsets of states such that transitions from one subset lead to the other, since it is possible to return to the starting state in two transitions. Thus, it cannot be possible to reach a state $i$ from a state $j$ in both an odd and an even number of transitions. However, if $m$ is odd, this is true for states 1 and $m$ – a contradiction (for example, see the case where $m = 5$ in Fig. 6.10, doors 1 and 5 can be reached from each other in 1 transition and also in 4 transitions).] The balance equations are

$$\begin{aligned} \pi_1 &= (1-p)\pi_2 + p\pi_m, \\ \pi_i &= p\pi_{i-1} + (1-p)\pi_{i+1}, \qquad i = 2, \ldots, m-1, \\ \pi_m &= (1-p)\pi_1 + p\pi_{m-1}. \end{aligned}$$

These equations are easily solved once we observe that by symmetry, all doors should have the same steady-state probability. This suggests the solution

$$\pi_j = \frac{1}{m}, \qquad j = 1, \ldots, m.$$

Indeed, we see that these $\pi_j$ satisfy the balance equations as well as the normalization equation, so they must be the desired steady-state probabilities (by the uniquenes part of the steady-state convergence theorem).

Note that if either $p = 0$ or $p = 1$, the chain still has a single recurrent class but is periodic. In this case, the $n$-step transition probabilities $r_{ij}(n)$ do not converge to a limit, because the doors are used in a cyclic order. Similarly, if $m$ is even, the recurrent class of the chain is periodic, since the states can be grouped into two subsets, the even and the odd numbered states, such that from each subset one can only go to the other subset.

**Example 6.7.** A machine can be either working or broken down on a given day. If it is working, it will break down in the next day with probability $b$, and will continue working with probability $1 - b$. If it breaks down on a given day, it will be repaired and be working in the next day with probability $r$, and will continue to be broken down with probability $1 - r$. What is the steady-state probability that the machine is working on a given day?

We introduce a Markov chain with the following two states:

State 1: Machine is working, State 2: Machine is broken down.

The transition probability graph of the chain is given in Fig. 6.11. The transition probability matrix is

$$\begin{bmatrix} 1-b & b \\ r & 1-r \end{bmatrix}.$$

This Markov chain has a single recurrent class that is aperiodic (assuming $0 < b < 1$ and $0 < r < 1$), and from the balance equations, we obtain

$$\pi_1 = (1-b)\pi_1 + r\pi_2, \qquad \pi_2 = b\pi_1 + (1-r)\pi_2,$$

or

$$b\pi_1 = r\pi_2.$$

This equation together with the normalization equation $\pi_1 + \pi_2 = 1$, yields the steady-state probabilities

$$\pi_1 = \frac{r}{b+r}, \qquad \pi_2 = \frac{b}{b+r}.$$

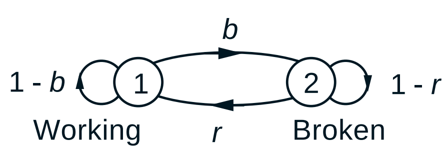



**Figure 6.11:** Transition probability graph for Example 6.7.

The situation considered in the previous example has evidently the Markov property, i.e., the state of the machine at the next day depends explicitly only on its state at the present day. However, it is possible to use a Markov chain model even if there is a dependence on the states at several past days. The general idea is to introduce some additional states which encode what has happened in preceding periods. Here is an illustration of this technique.

**Example 6.8.** Consider a variation of Example 6.7. If the machine remains broken for a given number of $\ell$ days, despite the repair efforts, it is replaced by a new working machine. To model this as a Markov chain, we replace the single state 2, corresponding to a broken down machine, with several states that indicate the number of days that the machine is broken. These states are

State $(2,i)$: The machine has been broken for $i$ days, $i = 1, 2, \ldots, \ell$.

The transition probability graph is given in Fig. 6.12 for the case where $\ell = 4$. Again this Markov chain has a single recurrent class that is aperiodic. From the balance equations, we have

$$\begin{aligned}\pi_1 &= (1-b)\pi_1 + r\big(\pi_{(2,1)} + \cdots + \pi_{(2,\ell-1)}\big) + \pi_{(2,\ell)},\\ \pi_{(2,1)} &= b\pi_1,\\ \pi_{(2,i)} &= (1-r)\pi_{(2,i-1)}, \qquad i = 2, \ldots, \ell.\end{aligned}$$

The last two equations can be used to express $\pi_{(2,i)}$ in terms of $\pi_1$,

$$\pi_{(2,i)} = (1-r)^{i-1} b\pi_1, \qquad i = 1, \ldots, \ell.$$

Substituting into the normalization equation $\pi_1 + \sum_{i=1}^{\ell} \pi_{(2,i)} = 1$, we obtain

$$1 = \left(1 + b\sum_{i=1}^{\ell}(1-r)^{i-1}\right)\pi_1 = \left(1 + \frac{b\big(1-(1-r)^{\ell}\big)}{r}\right)\pi_1,$$

or

$$\pi_1 = \frac{r}{r + b\big(1-(1-r)^{\ell}\big)}.$$

Using the equation $\pi_{(2,i)} = (1-r)^{i-1} b\pi_1$, we can also obtain explicit formulas for the $\pi_{(2,i)}$.

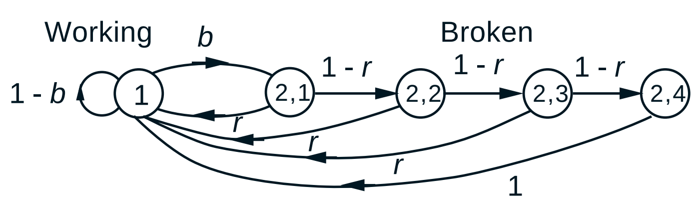



**Figure 6.12:** Transition probability graph for Example 6.8. A machine that has remained broken for $\ell = 4$ days is replaced by a new, working machine.

### Long-Term Frequency Interpretations

Probabilities are often interpreted as relative frequencies in an infinitely long string of independent trials. The steady-state probabilities of a Markov chain admit a similar interpretation, despite the absence of independence.

Consider, for example, a Markov chain involving a machine, which at the end of any day can be in one of two states, working or broken-down. Each time it breaks down, it is immediately repaired at a cost of \$1. How are we to model the long-term expected cost of repair **per day**? One possibility is to view it as the expected value of the repair cost on a randomly chosen day far into the future; this is just the steady-state probability of the broken down state. Alternatively, we can calculate the total expected repair cost in $n$ days, where $n$ is very large, and divide it by $n$. Intuition suggests that these two methods of calculation should give the same result. Theory supports this intuition, and in general we have the following interpretation of steady-state probabilities (a justification is given in the theoretical problems section).

**Steady-State Probabilities as Expected State Frequencies**

For a Markov chain with a single class that is aperiodic, the steady-state probabilities $\pi_j$ satisfy

$$\pi_j = \lim_{n\to\infty} \frac{v_{ij}(n)}{n},$$

where $v_{ij}(n)$ is the expected value of the number of visits to state $j$ within the first $n$ transitions, starting from state $i$.

Based on this interpretation, $\pi_j$ is the long-term expected fraction of time that the state is equal to $j$. Each time that state $j$ is visited, there is probability $p_{jk}$ that the next transition takes us to state $k$. We conclude that $\pi_j p_{jk}$ can be viewed as the long-term expected fraction of transitions that move the state from $j$ to $k$.†

† In fact, some stronger statements are also true. Namely, whenever we carry out the probabilistic experiment and generate a trajectory of the Markov chain over an infinite time horizon, the observed long-term frequency with which state $j$ is visited will be exactly equal to $\pi_j$, and the observed long-term frequency of transitions from $j$ to $k$ will be exactly equal to $\pi_j p_{jk}$. Even though the trajectory is random, these equalities hold with certainty, that is, with probability 1. The exact meaning of this statement will become more apparent in the next chapter, when we discuss concepts related to the limiting behavior of random processes.

**Expected Frequency of a Particular Transition**

Consider $n$ transitions of a Markov chain with a single class that is aperiodic, starting from a given initial state. Let $q_{jk}(n)$ be the expected number of such transitions that take the state from $j$ to $k$. Then, regardless of the initial state, we have

$$\lim_{n\to\infty} \frac{q_{jk}(n)}{n} = \pi_j p_{jk}.$$

The frequency interpretation of $\pi_j$ and $\pi_j p_{jk}$ allows for a simple interpretation of the balance equations. The state is equal to $j$ if and only if there is a transition that brings the state to $j$. Thus, the expected frequency $\pi_j$ of visits to $j$ is equal to the sum of the expected frequencies $\pi_k p_{kj}$ of transitions that lead to $j$, and

$$\pi_j = \sum_{k=1}^{m} \pi_k p_{kj};$$

see Fig. 6.13.

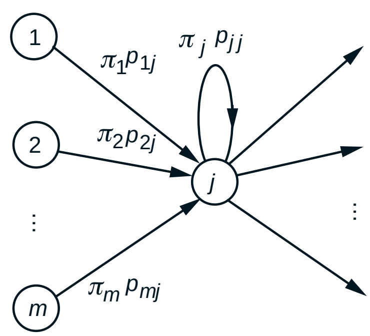



**Figure 6.13:** Interpretation of the balance equations in terms of frequencies. In a very large number of transitions, there will be a fraction $\pi_k p_{kj}$ that bring the state from $k$ to $j$. (This also applies to transitions from $j$ to itself, which occur with frequency $\pi_j p_{jj}$.) The sum of the frequencies of such transitions is the frequency $\pi_j$ of being at state $j$.

## Birth-Death Processes

A **birth-death** process is a Markov chain in which the states are linearly arranged and transitions can only occur to a neighboring state, or else leave the state unchanged. They arise in many contexts, especially in queueing theory.

Figure 6.14 shows the general structure of a birth-death process and also introduces some generic notation for the transition probabilities. In particular,

$$b_i = \mathbf{P}(X_{n+1} = i+1 \mid X_n = i), \qquad \text{("birth" probability at state } i\text{)},$$
$$d_i = \mathbf{P}(X_{n+1} = i-1 \mid X_n = i), \qquad \text{("death" probability at state } i\text{)}.$$

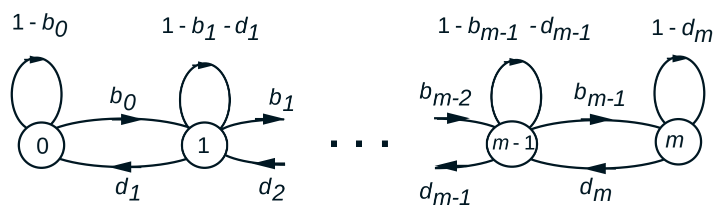



**Figure 6.14:** Transition probability graph for a birth-death process.

For a birth-death process, the balance equations can be substantially simplified. Let us focus on two neighboring states, say, $i$ and $i+1$. In any trajectory of the Markov chain, a transition from $i$ to $i+1$ has to be followed by a transition from $i+1$ to $i$, before another transition from $i$ to $i+1$ can occur. Therefore, the frequency of transitions from $i$ to $i+1$, which is $\pi_i b_i$, must be equal to the frequency of transitions from $i+1$ to $i$, which is $\pi_{i+1}d_{i+1}$. This leads to the **local balance** equations†

$$\pi_i b_i = \pi_{i+1} d_{i+1}, \qquad i = 0, 1, \ldots, m-1.$$

Using the local balance equations, we obtain

$$\pi_i = \pi_0 \frac{b_0 b_1 \cdots b_{i-1}}{d_1 d_2 \cdots d_i}, \qquad i = 1, \ldots, m.$$

Together with the normalization equation $\sum_i \pi_i = 1$, the steady-state probabilities $\pi_i$ are easily computed.

**Example 6.9. (Random Walk with Reflecting Barriers)** A person walks along a straight line and, at each time period, takes a step to the right with probability $b$, and a step to the left with probability $1-b$. The person starts in one of

† A more formal derivation that does not rely on the frequency interpretation proceeds as follows. The balance equation at state 0 is $\pi_0(1-b_0) + \pi_1 d_1 = \pi_0$, which yields the first local balance equation $\pi_0 b_0 = \pi_1 d_1$.

The balance equation at state 1 is $\pi_0 b_0 + \pi_1(1-b_1-d_1) + \pi_2 d_2 = \pi_1$. Using the local balance equation $\pi_0 b_0 = \pi_1 d_1$ at the previous state, this is rewritten as $\pi_1 d_1 + \pi_1(1-b_1-d_1) + \pi_2 d_2 = \pi_1$, which simplifies to $\pi_1 b_1 = \pi_2 d_2$. We can then continue similarly to obtain the local balance states at all other states.

the positions $1, 2, \ldots, m$, but if he reaches position 0 (or position $m+1$), his step is instantly reflected back to position 1 (or position $m$, respectively). Equivalently, we may assume that when the person is in positions 1 or $m$. he will stay in that position with corresponding probability $1-b$ and $b$, respectively. We introduce a Markov chain model whose states are the positions $1, \ldots, m$. The transition probability graph of the chain is given in Fig. 6.15.

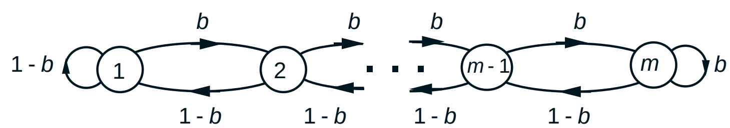



**Figure 6.15:** Transition probability graph for the random walk Example 6.9.

The local balance equations are

$$\pi_i b = \pi_{i+1}(1-b), \qquad i = 1, \ldots, m-1.$$

Thus, $\pi_{i+1} = \rho\pi_i$, where

$$\rho = \frac{b}{1-b},$$

and we can express all the $\pi_j$ in terms of $\pi_1$, as

$$\pi_i = \rho^{i-1}\pi_1, \qquad i = 1, \ldots, m.$$

Using the normalization equation $1 = \pi_1 + \cdots + \pi_m$, we obtain

$$1 = \pi_1(1 + \rho + \cdots + \rho^{m-1})$$

which leads to

$$\pi_i = \frac{\rho^{i-1}}{1 + \rho + \cdots + \rho^{m-1}}, \qquad i = 1, \ldots, m.$$

Note that if $\rho = 1$, then $\pi_i = 1/m$ for all $i$.

**Example 6.10. (Birth-Death Markov Chains – Queueing)** Packets arrive at a node of a communication network, where they are stored in a buffer and then transmitted. The storage capacity of the buffer is $m$: if $m$ packets are already present, any newly arriving packets are discarded. We discretize time in very small periods, and we assume that in each period, at most one event can happen that can change the number of packets stored in the node (an arrival of a new packet or a completion of the transmission of an existing packet). In particular, we assume that at each period, exactly one of the following occurs:

(a) one new packet arrives; this happens with a given probability $b > 0$;

(b) one existing packet completes transmission; this happens with a given probability $d > 0$ if there is at least one packet in the node, and with probability 0 otherwise;

(c) no new packet arrives and no existing packet completes transmission; this happens with a probability $1-b-d$ if there is at least one packet in the node, and with probability $1-b$ otherwise.

We introduce a Markov chain with states $0, 1, \ldots, m$, corresponding to the number of packets in the buffer. The transition probability graph is given in Fig. 6.16.

The local balance equations are

$$\pi_i b = \pi_{i+1} d, \qquad i = 0, 1, \ldots, m-1.$$

We define

$$\rho = \frac{b}{d},$$

and obtain $\pi_{i+1} = \rho \pi_i$, which leads to $\pi_i = \rho^i \pi_0$ for all $i$. By using the normalization equation $1 = \pi_0 + \pi_1 + \cdots + \pi_m$, we obtain

$$1 = \pi_0(1 + \rho + \cdots + \rho^m),$$

and

$$\pi_0 = \begin{cases} \dfrac{1-\rho}{1-\rho^{m+1}} & \text{if } \rho \neq 1, \\ \dfrac{1}{m+1} & \text{if } \rho = 1. \end{cases}$$

The steady-state probabilities are then given by

$$\pi_i = \begin{cases} \dfrac{\rho^i(1-\rho)}{1-\rho^{m+1}} & \text{if } \rho \neq 1, \\ \dfrac{1}{m+1} & \text{if } \rho = 1, \end{cases} \qquad i = 0, 1, \ldots, m.$$

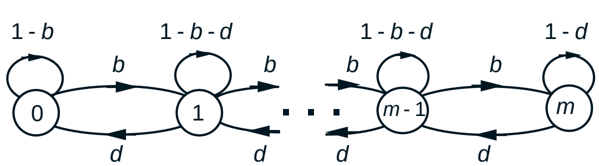



**Figure 6.16:** Transition probability graph in Example 6.10.

It is interesting to consider what happens when the buffer size $m$ is so large that it can be considered as practically infinite. We distinguish two cases.

(a) Suppose that $b < d$, or $\rho < 1$. In this case, arrivals of new packets are less likely than departures of existing packets. This prevents the number of packets in the buffer from growing, and the steady-state probabilities $\pi_i$ decrease with $i$. We observe that as $m \to \infty$, we have $1 - \rho^{m+1} \to 1$, and

$$\pi_i \to \rho^i(1-\rho), \qquad \text{for all } i.$$

We can view these as the steady-state probabilities in a system with an infinite buffer. [As a check, note that we have $\sum_{i=0}^{\infty} \rho^i(1-\rho) = 1$.]

(b) Suppose that $b > d$, or $\rho > 1$. In this case, arrivals of new packets are more likely than departures of existing packets. The number of packets in the buffer tends to increase, and the steady-state probabilities $\pi_i$ increase with $i$. As we consider larger and larger buffer sizes $m$, the steady-state probability of any fixed state $i$ decreases to zero:

$$\pi_i \to 0, \qquad \text{for all } i.$$

Were we to consider a system with an infinite buffer, we would have a Markov chain with a countably infinite number of states. Although we do not have the machinery to study such chains, the preceding calculation suggests that every state will have zero steady-state probability and will be "transient." The number of packets in queue will generally grow to infinity, and any particular state will be visited only a finite number of times.

## 6.4 ABSORPTION PROBABILITIES AND EXPECTED TIME TO ABSORPTION

In this section, we study the short-term behavior of Markov chains. We first consider the case where the Markov chain starts at a transient state. We are interested in the first recurrent state to be entered, as well as in the time until this happens.

When focusing on such questions, the subsequent behavior of the Markov chain (after a recurrent state is encountered) is immaterial. We can therefore assume, without loss of generality, that every recurrent state $k$ is **absorbing**, i.e.,

$$p_{kk} = 1, \qquad p_{kj} = 0 \ \text{ for all } j \neq k.$$

If there is a unique absorbing state $k$, its steady-state probability is 1 (because all other states are transient and have zero steady-state probability), and will be reached with probability 1, starting from any initial state. If there are multiple absorbing states, the probability that one of them will be eventually reached is still 1, but the identity of the absorbing state to be entered is random and the associated probabilities may depend on the starting state. In the sequel, we fix a particular absorbing state, denoted by $s$, and consider the absorption probability $a_i$ that $s$ is eventually reached, starting from $i$:

$$a_i = \mathbf{P}(X_n \text{ eventually becomes equal to the absorbing state } s \,|\, X_0 = i).$$

Absorption probabilities can be obtained by solving a system of linear equations, as indicated below.

**Absorption Probability Equations**

Consider a Markov chain in which each state is either transient or absorbing. We fix a particular absorbing state $s$. Then, the probabilities $a_i$ of eventually reaching state $s$, starting from $i$, are the unique solution of the equations

$$\begin{aligned} a_s &= 1, \\ a_i &= 0, \qquad \text{for all absorbing } i \neq s, \\ a_i &= \sum_{j=1}^{m} p_{ij} a_j, \qquad \text{for all transient } i. \end{aligned}$$

The equations $a_s = 1$, and $a_i = 0$, for all absorbing $i \neq s$, are evident from the definitions. To verify the remaining equations, we argue as follows. Let us consider a transient state $i$ and let $A$ be the event that state $s$ is eventually reached. We have

$$\begin{aligned} a_i &= \mathbf{P}(A \,|\, X_0 = i) \\ &= \sum_{j=1}^{m} \mathbf{P}(A \,|\, X_0 = i, X_1 = j)\mathbf{P}(X_1 = j \,|\, X_0 = i) \qquad \text{(total probability thm.)} \\ &= \sum_{j=1}^{m} \mathbf{P}(A \,|\, X_1 = j) p_{ij} \qquad \text{(Markov property)} \\ &= \sum_{j=1}^{m} a_j p_{ij}. \end{aligned}$$

The uniqueness property of the solution of the absorption probability equations requires a separate argument, which is given in the theoretical problems section.

The next example illustrates how we can use the preceding method to calculate the probability of entering a given recurrent class (rather than a given absorbing state).

**Example 6.11.** Consider the Markov chain shown in Fig. 6.17(a). We would like to calculate the probability that the state eventually enters the recurrent class

$\{4, 5\}$ starting from one of the transient states. For the purposes of this problem, the possible transitions within the recurrent class $\{4, 5\}$ are immaterial. We can therefore lump the states in this recurrent class and treat them as a single absorbing state (call it state 6); see Fig. 6.17(b). It then suffices to compute the probability of eventually entering state 6 in this new chain.

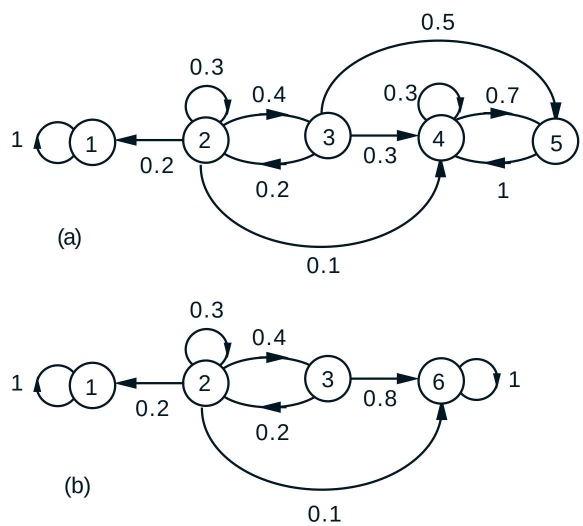



**Figure 6.17:** (a) Transition probability graph in Example 6.11. (b) A new graph in which states 4 and 5 have been lumped into the absorbing state $s = 6$.

The absorption probabilities $a_i$ of eventually reaching state $s = 6$ starting from state $i$, satisfy the following equations:

$$a_2 = 0.2a_1 + 0.3a_2 + 0.4a_3 + 0.1a_6,$$
$$a_3 = 0.2a_2 + 0.8a_6.$$

Using the facts $a_1 = 0$ and $a_6 = 1$, we obtain

$$a_2 = 0.3a_2 + 0.4a_3 + 0.1,$$
$$a_3 = 0.2a_2 + 0.8.$$

This is a system of two equations in the two unknowns $a_2$ and $a_3$, which can be readily solved to yield $a_2 = 21/31$ and $a_3 = 29/31$.

**Example 6.12. (Gambler's Ruin)** A gambler wins \$1 at each round, with probability $p$, and loses \$1, with probability $1 - p$. Different rounds are assumed independent. The gambler plays continuously until he either accumulates a target amount of $\$m$, or loses all his money. What is the probability of eventually accumulating the target amount (winning) or of losing his fortune?

We introduce the Markov chain shown in Fig. 6.18 whose state $i$ represents the gambler's wealth at the beginning of a round. The states $i = 0$ and $i = m$ correspond to losing and winning, respectively.

All states are transient, except for the winning and losing states which are absorbing. Thus, the problem amounts to finding the probabilities of absorption at each one of these two absorbing states. Of course, these absorption probabilities depend on the initial state $i$.

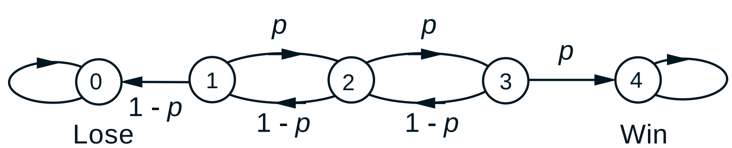



**Figure 6.18:** Transition probability graph for the gambler's ruin problem (Example 6.12). Here $m = 4$.

Let us set $s = 0$ in which case the absorption probability $a_i$ is the probability of losing, starting from state $i$. These probabilities satisfy

$$\begin{aligned} a_0 &= 1, \\ a_i &= (1-p)a_{i-1} + pa_{i+1}, \qquad i = 1, \ldots, m-1, \\ a_m &= 0. \end{aligned}$$

These equations can be solved in a variety of ways. It turns out there is an elegant method that leads to a nice closed form solution.

Let us write the equations for the $a_i$ as

$$(1-p)(a_{i-1} - a_i) = p(a_i - a_{i+1}), \qquad i = 1, \ldots, m-1.$$

Then, by denoting

$$\delta_i = a_i - a_{i+1}, \qquad i = 1, \ldots, m-1,$$

and

$$\rho = \frac{1-p}{p},$$

the equations are written as

$$\delta_i = \rho\delta_{i-1}, \qquad i = 1, \ldots, m-1,$$

from which we obtain

$$\delta_i = \rho^i\delta_0, \qquad i = 1, \ldots, m-1.$$

This, together with the equation $\delta_0 + \delta_1 + \cdots + \delta_{m-1} = a_0 - a_m = 1$, implies that

$$(1 + \rho + \cdots + \rho^{m-1})\delta_0 = 1.$$

Thus, we have

$$\delta_0 = \begin{cases} \dfrac{1-\rho}{1-\rho^m} & \text{if } \rho \neq 1, \\ \dfrac{1}{m} & \text{if } \rho = 1, \end{cases}$$

and, more generally,

$$\delta_i = \begin{cases} \dfrac{\rho^i(1-\rho)}{1-\rho^m} & \text{if } \rho \neq 1, \\ \dfrac{1}{m} & \text{if } \rho = 1. \end{cases}$$

From this relation, we can calculate the probabilities $a_i$. If $\rho \neq 1$, we have

$$\begin{aligned} a_i &= a_0 - \delta_{i-1} - \cdots - \delta_0 \\ &= 1 - (\rho^{i-1} + \cdots + \rho + 1)\delta_0 \\ &= 1 - \frac{1-\rho^i}{1-\rho} \cdot \frac{1-\rho}{1-\rho^m}, \\ &= 1 - \frac{1-\rho^i}{1-\rho^m}, \end{aligned}$$

and finally the probability of losing, starting from a fortune $i$, is

$$a_i = \frac{\rho^i - \rho^m}{1-\rho^m}, \qquad i = 1, \ldots, m-1.$$

If $\rho = 1$, we similarly obtain

$$a_i = \frac{m-i}{m}.$$

The probability of winning, starting from a fortune $i$, is the complement $1-a_i$, and is equal to

$$1 - a_i = \begin{cases} \dfrac{1-\rho^i}{1-\rho^m} & \text{if } \rho \neq 1, \\ \dfrac{i}{m} & \text{if } \rho = 1. \end{cases}$$

The solution reveals that if $\rho > 1$, which corresponds to $p < 1/2$ and unfavorable odds for the gambler, the probability of losing approaches 1 as $m \to \infty$ regardless of the size of the initial fortune. This suggests that if you aim for a large profit under unfavorable odds, financial ruin is almost certain.

### Expected Time to Absorption

We now turn our attention to the expected number of steps until a recurrent state is entered (an event that we refer to as "absorption"), starting from a particular transient state. For any state $i$, we denote

$$\begin{aligned}\mu_i &= \mathbf{E}\big[\text{number of transitions until absorption, starting from } i\big]\\ &= \mathbf{E}\big[\min\{n \geq 0 \,|\, X_n \text{ is recurrent}\} \;\big|\; X_0 = i\big].\end{aligned}$$

If $i$ is recurrent, this definition sets $\mu_i$ to zero.

We can derive equations for the $\mu_i$ by using the total expectation theorem. We argue that the time to absorption starting from a transient state $i$ is equal to 1 plus the expected time to absorption starting from the next state, which is $j$ with probability $p_{ij}$. This leads to a system of linear equations which is stated below. It turns out that these equations have a unique solution, but the argument for establishing this fact is beyond our scope.

**Equations for the Expected Time to Absorption**

The expected times $\mu_i$ to absorption, starting from state $i$ are the unique solution of the equations

$$\mu_i = 0, \qquad \text{for all recurrent states } i,$$

$$\mu_i = 1 + \sum_{j=1}^{m} p_{ij}\mu_j, \qquad \text{for all transient states } i.$$

**Example 6.13. (Spiders and Fly)** Consider the spiders-and-fly model of Example 6.2. This corresponds to the Markov chain shown in Fig. 6.19. The states correspond to possible fly positions, and the absorbing states 1 and $m$ correspond to capture by a spider.

Let us calculate the expected number of steps until the fly is captured. We have

$$\mu_1 = \mu_m = 0,$$

and

$$\mu_i = 1 + 0.3 \cdot \mu_{i-1} + 0.4 \cdot \mu_i + 0.3 \cdot \mu_{i+1}, \qquad \text{for } i = 2, \ldots, m-1.$$

We can solve these equations in a variety of ways, such as for example by successive substitution. As an illustration, let $m = 4$, in which case, the equations reduce to

$$\mu_2 = 1 + 0.4 \cdot \mu_2 + 0.3 \cdot \mu_3, \qquad \mu_3 = 1 + 0.3 \cdot \mu_2 + 0.4 \cdot \mu_3.$$

The first equation yields $\mu_2 = (1/0.6) + (1/2)\mu_3$, which we can substitute in the second equation and solve for $\mu_3$. We obtain $\mu_3 = 10/3$ and by substitution again, $\mu_2 = 10/3$.

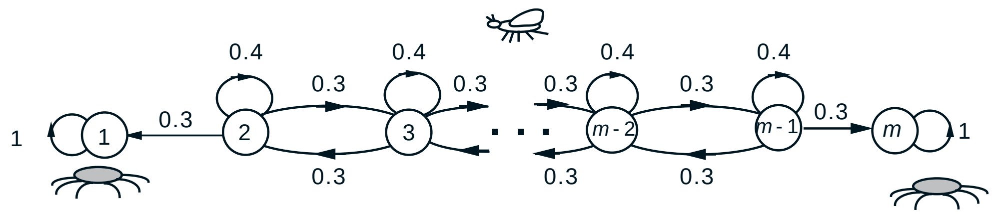



**Figure 6.19:** Transition probability graph in Example 6.13.

### Mean First Passage Times

The same idea used to calculate the expected time to absorption can be used to calculate the expected time to reach a particular recurrent state, starting from any other state. Throughout this subsection, we consider a Markov chain with a single recurrent class. We focus on a special recurrent state $s$, and we denote by $t_i$ the **mean first passage time from state** $i$ **to state** $s$, defined by

$$\begin{aligned} t_i &= \mathbf{E}\big[\text{number of transitions to reach } s \text{ for the first time, starting from } i\big] \\ &= \mathbf{E}\big[\min\{n \geq 0 \,|\, X_n = s\} \;\big|\; X_0 = i\big]. \end{aligned}$$

The transitions out of state $s$ are irrelevant to the calculation of the mean first passage times. We may thus consider a new Markov chain which is identical to the original, except that the special state $s$ is converted into an absorbing state (by setting $p_{ss} = 1$, and $p_{sj} = 0$ for all $j \neq s$). We then compute $t_i$ as the expected number of steps to absorption starting from $i$, using the formulas given earlier in this section. We have

$$\begin{aligned} t_i &= 1 + \sum_{j=1}^{m} p_{ij} t_j, \qquad \text{for all } i \neq s, \\ t_s &= 0. \end{aligned}$$

This system of linear equations can be solved for the unknowns $t_i$, and is known to have a unique solution.

The above equations give the expected time to reach the special state $s$ starting from any other state. We may also want to calculate the **mean recurrence time** of the special state $s$, which is defined as

$$\begin{aligned} t_s^* &= \mathbf{E}[\text{number of transitions up to the first return to } s, \text{ starting from } s] \\ &= \mathbf{E}\big[\min\{n > 1 \,|\, X_n = s\} \;\big|\; X_0 = s\big]. \end{aligned}$$

We can obtain $t_s^*$, once we have the first passage times $t_i$, by using the equation

$$t_s^* = 1 + \sum_{j=1}^{m} p_{sj} t_j.$$

To justify this equation, we argue that the time to return to $s$, starting from $s$, is equal to 1 plus the expected time to reach $s$ from the next state, which is $j$ with probability $p_{sj}$. We then apply the total expectation theorem.

**Example 6.14.** Consider the "up-to-date"–"behind" model of Example 6.1. States 1 and 2 correspond to being up-to-date and being behind, respectively, and the transition probabilities are

$$p_{11} = 0.8, \qquad p_{12} = 0.2,$$
$$p_{21} = 0.6, \qquad p_{22} = 0.4.$$

Let us focus on state $s = 1$ and calculate the mean first passage time to state 1, starting from state 2. We have $t_1 = 0$ and

$$t_2 = 1 + p_{21}t_1 + p_{22}t_2 = 1 + 0.4 \cdot t_2,$$

from which

$$t_2 = \frac{1}{0.6} = \frac{5}{3}.$$

The mean recurrence time to state 1 is given by

$$t_1^* = 1 + p_{11}t_1 + p_{12}t_2 = 1 + 0 + 0.2 \cdot \frac{5}{3} = \frac{4}{3}.$$

**Summary of Facts About Mean First Passage Times**

Consider a Markov chain with a single recurrent class, and let $s$ be a particular recurrent state.

- The mean first passage times $t_i$ to reach state $s$ starting from $i$, are the unique solution to the system of equations

$$t_s = 0, \qquad t_i = 1 + \sum_{j=1}^{m} p_{ij} t_j, \qquad \text{for all } i \neq s.$$

- The mean recurrence time $t_s^*$ of state $s$ is given by

$$t_s^* = 1 + \sum_{j=1}^{m} p_{sj} t_j.$$

## 6.5 MORE GENERAL MARKOV CHAINS

The discrete-time, finite-state Markov chain model that we have considered so far is the simplest example of an important Markov process. In this section, we briefly discuss some generalizations that involve either a countably infinite number of states or a continuous time, or both. A detailed theoretical development for these types of models is beyond our scope, so we just discuss their main underlying ideas, relying primarily on examples.

### Chains with Countably Infinite Number of States

Consider a Markov process $\{X_1, X_2, \ldots\}$ whose state can take any positive integer value. The transition probabilities

$$p_{ij} = \mathbf{P}(X_{n+1} = j \mid X_n = i), \qquad i, j = 1, 2, \ldots$$

are given, and can be used to represent the process with a transition probability graph that has an infinite number of nodes, corresponding to the integers $1, 2, \ldots$

It is straightforward to verify, using the total probability theorem in a similar way as in Section 6.1, that the $n$-step transition probabilities

$$r_{ij}(n) = \mathbf{P}(X_n = j \mid X_0 = i), \qquad i, j = 1, 2, \ldots$$

satisfy the Chapman-Kolmogorov equations

$$r_{ij}(n+1) = \sum_{k=1}^{\infty} r_{ik}(n) p_{kj}, \qquad i, j = 1, 2, \ldots$$

Furthermore, if the $r_{ij}(n)$ converge to steady-state values $\pi_j$ as $n \to \infty$, then by taking limit in the preceding equation, we obtain

$$\pi_j = \sum_{k=1}^{\infty} \pi_k p_{kj}, \qquad i, j = 1, 2, \ldots$$

These are the balance equations for a Markov chain with states $1, 2, \ldots$

It is important to have conditions guaranteeing that the $r_{ij}(n)$ indeed converge to steady-state values $\pi_j$ as $n \to \infty$. As we can expect from the finite-state case, such conditions should include some analog of the requirement that there is a single recurrent class that is aperiodic. Indeed, we require that:

(a) each state is accessible from every other state;

(b) the set of all states is aperiodic in the sense that there is no $d > 1$ such that the states can be grouped in $d > 1$ disjoint subsets $S_1, \ldots, S_d$ so that all transitions from one subset lead to the next subset.

These conditions are sufficient to guarantee the convergence to a steady-state

$$\lim_{n\to\infty} r_{ij}(n) = \pi_j, \qquad i, j = 1, 2, \ldots$$

but something peculiar may also happen here, which is not possible if the number of states is finite: the limits $\pi_j$ may not add to 1, so that $(\pi_1, \pi_2, \ldots)$ may not be a probability distribution. In fact, we can prove the following theorem (the proof is beyond our scope).

**Steady-State Convergence Theorem**

Under the above accessibility and aperiodicity assumptions (a) and (b), there are only two possibilities:

(1) The $r_{ij}(n)$ converge to a steady state probability distribution $(\pi_1, \pi_2, \ldots)$. In this case the $\pi_j$ uniquely solve the balance equations together with the normalization equation $\pi_1 + \pi_2 + \cdots = 1$. Furthermore, the $\pi_j$ have an expected frequency interpretation:

$$\pi_j = \lim_{n\to\infty} \frac{v_{ij}(n)}{n},$$

where $v_{ij}(n)$ is the expected number of visits to state $j$ within the first $n$ transitions, starting from state $i$.

(2) All the $r_{ij}(n)$ converge to 0 as $n \to \infty$ and the balance equations have no solution, other than $\pi_j = 0$ for all $j$.

For an example of possibility (2) above, consider the packet queueing system of Example 6.10 for the case where the probability $b$ of a packet arrival in each period is larger than the probability $d$ of a departure. Then, as we saw in that example, as the buffer size $m$ increases, the size of the queue will tend to increase without bound, and the steady-state probability of any one state will tend to 0 as $m \to \infty$. In effect, with infinite buffer space, the system is "unstable" when $b > d$, and all states are "transient."

An important consequence of the steady-state convergence theorem is that if we can find a probability distribution $(\pi_1, \pi_2, \ldots)$ that solves the balance equations, then we can be sure that it is the steady-state distribution. This line of argument is very useful in queueing systems as illustrated in the following two examples.

**Example 6.15. (Queueing with Infinite Buffer Space)** Consider, as in Example 6.10, a communication node, where packets arrive and are stored in a buffer before getting transmitted. We assume that the node can store an infinite number of packets. We discretize time in very small periods, and we assume that in each period, one of the following occurs:

(a) one new packet arrives; this happens with a given probability $b > 0$;

(b) one existing packet completes transmission; this happens with a given probability $d > 0$ if there is at least one packet in the node, and with probability 0 otherwise;

(c) no new packet arrives and no existing packet completes transmission; this happens with a probability $1-b-d$ if there is at least one packet in the node, and with probability $1-b$ otherwise.

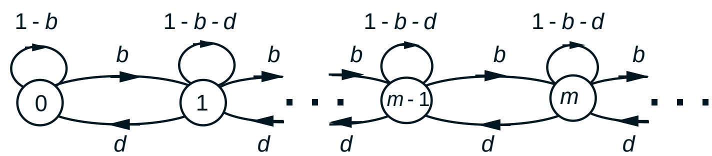



**Figure 6.20:** Transition probability graph in Example 6.15.

We introduce a Markov chain with states are $0, 1, \ldots$, corresponding to the number of packets in the buffer. The transition probability graph is given in Fig. 6.20. As in the case of a finite number of states, the local balance equations are

$$\pi_i b = \pi_{i+1} d, \qquad i = 0, 1, \ldots,$$

and we obtain $\pi_{i+1} = \rho \pi_i$, where $\rho = b/d$. Thus, we have $\pi_i = \rho^i \pi_0$ for all $i$. If $\rho < 1$, the normalization equation $1 = \sum_{i=0}^{\infty} \pi_i$ yields

$$1 = \pi_0 \sum_{i=0}^{\infty} \rho^i = \frac{\pi_0}{1-\rho},$$

in which case $\pi_0 = 1 - \rho$, and the steady-state probabilities are

$$\pi_i = \rho^i (1-\rho), \qquad i = 0, 1, \ldots$$

If $\rho \geq 1$, which corresponds to the case where the arrival probability $b$ is no less than the departure probability $d$, the normalization equation $1 = \pi_0(1+\rho+\rho^2+\cdots)$ implies that $\pi_0 = 0$, and also $\pi_i = \rho^i \pi_0 = 0$ for all $i$.

**Example 6.16. (The $M/G/1$ Queue)** Packets arrive at a node of a communication network, where they are stored at an infinite capacity buffer and are then transmitted one at a time. The arrival process of the packets is Poissson with rate

$\lambda$, and the transmission time of a packet has a given CDF. Furthermore, the transmission times of different packets are independent and are also independent from all the interarrival times of the arrival process.

This queueing system is known as the $M/G/1$ system. With changes in terminology, it applies to many different practical contexts where "service" is provided to "arriving customers," such as in communication, transportation, and manufacturing, among others. The name $M/G/1$ is an example of shorthand terminology from queueing theory, whereby the first letter ($M$ in this case) characterizes the customer arrival process (Poisson in this case), the second letter ($G$ in this case) characterizes the distribution of the service time of the queue (general in this case), and the number (1 in this case) characterizes the number of customers that can be simultaneously served.

To model this system as a discrete-time Markov chain, we focus on the time instants when a packet completes transmission and departs from the system. We denote by $X_n$ the number of packets in the system just after the $n$th customer's departure. We have

$$X_{n+1} = \begin{cases} X_n - 1 + S_n & \text{if } X_n > 0, \\ S_n & \text{if } X_n = 0, \end{cases}$$

where $S_n$ is the number of packet arrivals during the $(n+1)$st packet's transmission. In view of the Poisson assumption, the random variables $S_1, S_2, \ldots$ are independent and their PMF can be calculated using the given CDF of the transmission time, and the fact that in an interval of length $r$, the number of packet arrivals is Poisson-distributed with parameter $\lambda r$. In particular, let us denote

$$\alpha_k = \mathbf{P}(S_n = k), \qquad k = 0, 1, \ldots,$$

and let us assume that if the transmission time $R$ of a packet is a discrete random variable taking the values $r_1, \ldots, r_m$ with probabilities $p_1, \ldots, p_m$. Then, we have for all $k \geq 0$,

$$\alpha_k = \sum_{j=1}^{m} p_j \frac{e^{-\lambda r_j}(\lambda r_j)^k}{k!},$$

while if $R$ is a continuous random variable with PDF $f_R(r)$, we have for all $k \geq 0$,

$$\alpha_k = \int_{r=0}^{\infty} \mathbf{P}(S_n = k \mid R = r) f_R(r)\, dr = \int_{r=0}^{\infty} \frac{e^{-\lambda r}(\lambda r)^k}{k!} f_R(r)\, dr.$$

The probabilities $\alpha_k$ define in turn the transition probabilities of the Markov chain $\{X_n\}$, as follows (see Fig. 6.21):

$$p_{ij} = \begin{cases} \alpha_j & \text{if } i = 0 \text{ and } j > 0, \\ \alpha_{j-i+1} & \text{if } i > 0 \text{ and } j \geq i - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, this Markov chain satisfies the accessibility and aperiodicity conditions that guarantee steady-state convergence. There are two possibilities: either $(\pi_0, \pi_1, \ldots)$ form a probability distribution, or else $\pi_j > 0$ for all $j$. We will clarify

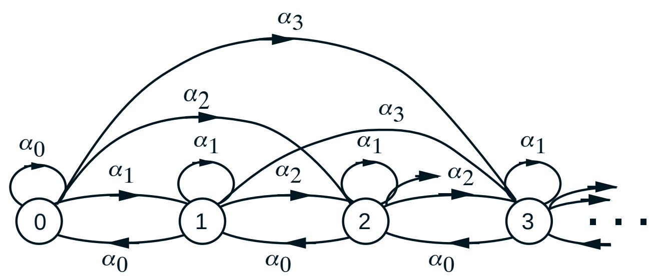



**Figure 6.21:** Transition probability graph for the number of packets left behind by a packet completing transmission in the $M/G/1$ queue (Example 6.16).

the conditions under which each of these cases holds, and we will also calculate the transform $M(s)$ (when it exists) of the steady-state distribution $(\pi_0, \pi_1, \ldots)$:

$$M(s) = \sum_{j=0}^{\infty} \pi_j e^{sj}.$$

For this purpose, we will use the transform of the PMF $\{\alpha_k\}$:

$$A(s) = \sum_{j=0}^{\infty} \alpha_j e^{sj}.$$

Indeed, let us multiply the balance equations

$$\pi_j = \pi_0 \alpha_j + \sum_{i=1}^{j+1} \pi_i \alpha_{j-i+1},$$

with $e^{sj}$ and add over all $j$. We obtain

$$\begin{aligned} M(s) &= \sum_{j=0}^{\infty} \pi_0 \alpha_j e^{sj} + \sum_{j=0}^{\infty} \left( \sum_{i=1}^{j+1} \pi_i \alpha_{j-i+1} \right) e^{sj} \\ &= A(s) + \sum_{i=1}^{\infty} \pi_i e^{s(i-1)} \sum_{j=i-1}^{\infty} \alpha_{j-i+1} e^{s(j-i+1)} \\ &= A(s) + \frac{A(s)}{e^s} \sum_{i=1}^{\infty} \pi_i e^{si} \\ &= A(s) + \frac{A(s)\big(M(s) - \pi_0\big)}{e^s}, \end{aligned}$$

or

$$M(s) = \frac{(e^s - 1)\pi_0 A(s)}{e^s - A(s)}.$$

To calculate $\pi_0$, we take the limit as $s \to 0$ in the above formula, and we use the fact $M(0) = 1$ when $\{\pi_j\}$ is a probability distribution. We obtain, using the fact $A(0) = 1$ and L'Hospital's rule,

$$1 = \lim_{s \to 0} \frac{(e^s - 1)\pi_0 A(s)}{e^s - A(s)} = \frac{\pi_0}{1 - \big(dA(s)/ds\big)\big|_{s=0}} = \frac{\pi_0}{1 - \mathbf{E}[N]},$$

where $\mathbf{E}[N] = \sum_{j=0}^{\infty} j\alpha_j$ is the expected value of the number $N$ of packet arrivals within a packet's transmission time. Using the iterated expectations formula, we have

$$\mathbf{E}[N] = \lambda \mathbf{E}[R],$$

where $\mathbf{E}[R]$ is the expected value of the transmission time. Thus,

$$\pi_0 = 1 - \lambda \mathbf{E}[R],$$

and the transform of the steady-state distribution $\{\pi_j\}$ is

$$M(s) = \frac{(e^s - 1)\big(1 - \lambda \mathbf{E}[R]\big) A(s)}{e^s - A(s)}.$$

For the above calculation to be correct, we must have $\mathbf{E}[N] < 1$, i.e., packets should arrive at a rate that is smaller than the transmission rate of the node. If this is not true, the system is not "stable" and there is no steady-state distribution, i.e., the only solution of the balance equations is $\pi_j = 0$ for all $j$.

Let us finally note that we have introduced the $\pi_j$ as the steady-state probability that $j$ packets are left behind in the system by a packet upon completing transmission. However, it turns out that $\pi_j$ is also equal to the steady-state probability of $j$ packets found in the system by an observer that looks at the system at a "typical" time far into the future. This is discussed in the theoretical problems, but to get an idea of the underlying reason, note that for each time the number of packets in the system increases from $n$ to $n + 1$ due to an arrival, there will be a corresponding future decrease from $n + 1$ to $n$ due to a departure. Therefore, in the long run, the frequency of transitions from $n$ to $n + 1$ is equal to the frequency of transitions from $n + 1$ to $n$. Therefore, in steady-state, the system appears statistically identical to an arriving and to a departing packet. Now, because the packet interarrival times are independent and exponentially distributed, the times of packet arrivals are "typical" and do not depend on the number of packets in the system. With some care this argument can be made precise, and shows that at the times when packets complete their transmissions and depart, the system is "typically loaded."

## Continuous-Time Markov Chains

We have implicitly assumed so far that the transitions between states take unit time. When the time between transitions takes values from a continuous range, some new questions arise. For example, what is the proportion of time that the system spends at a particular state (as opposed to the frequency of visits into the state)?

Let the states be denoted by $1, 2, \ldots$, and let us assume that state transitions occur at discrete times, but the time from one transition to the next is random. In particular, we assume that:

(a) If the current state is $i$, the next state will be $j$ with a given probability $p_{ij}$.

(b) The time interval $\Delta_i$ between the transition to state $i$ and the transition to the next state is exponentially distributed with a given parameter $\nu_i$:

$$\mathbf{P}(\Delta_i \leq \delta \,|\, \text{current state is } i) \leq 1 - e^{-\nu_i \delta}.$$

Furthermore, $\Delta_i$ is independent of earlier transition times and states.

The parameter $\nu_i$ is referred to as the *transition rate associated with state* $i$. Since the expected transition time is

$$\mathbf{E}[\Delta_i] = \int_0^\infty \delta \nu_i e^{-\nu_i \delta} d\delta = \frac{1}{\nu_i},$$

we can interpret $\nu_i$ as the average number of transitions per unit time. We may also view

$$q_{ij} = p_{ij}\nu_i$$

as the rate at which the process makes a transition to $j$ when at state $i$. Consequently, we call $q_{ij}$ the *transition rate from* $i$ *to* $j$. Note that given the transition rates $q_{ij}$, one can obtain the node transition rates using the formula $\nu_i = \sum_{j=1}^{\infty} q_{ij}$.

The state of the chain at time $t \geq 0$ is denoted by $X(t)$, and stays constant between transitions. Let us recall the memoryless property of the exponential distribution, which in our context implies that, for any time $t$ between the $k$th and $(k+1)$st transition times $t_k$ and $t_{k+1}$, the additional time $t_{k+1} - t$ needed to effect the next transition is independent of the time $t - t_k$ that the system has been in the current state. This implies the Markov character of the process, i.e., that at any time $\overline{t}$, the future of the process, [the random variables $X(t)$ for $t > \overline{t}$] depend on the past of the process [the values of the random variables $X(t)$ for $t \leq \overline{t}$] only through the present value of $X(\overline{t})$.

**Example 6.17. (The M/M/1 Queue)** Packets arrive at a node of a communication network according to a Poissson process with rate $\lambda$. The packets are stored at an infinite capacity buffer and are then transmitted one at a time. The transmission time of a packet is exponentially distributed with parameter $\mu$, and the transmission times of different packets are independent and are also independent from all the interarrival times of the arrival process. Thus, this queueing system is identical to the special case of the $M/G/1$ system, where the transmission times are exponentially distributed (this is indicated by the second $M$ inthe $M/M/1$ name).

We will model this system using a continuous-time process with state $X(t)$ equal to the number of packets in the system at time $t$ [if $X(t) > 0$, then $X(t) - 1$ packets are waiting in the queue and one packet is under transmission]. The state increases by one when a new packet arrives and decreases by one when an existing packet departs. To show that this process is a continuous-time Markov chain, let us identify the transition rates $\nu_i$ and $q_{ij}$ at each state $i$.

Consider first the case where at some time $\bar{t}$, the system becomes empty, i.e., the state becomes equal to 0. Then the next transition will occur at the next arrival, which will happen in time that is exponentially distributed with parameter $\lambda$. Thus at state 0, we have the transition rates

$$q_{0j} = \begin{cases} \lambda & \text{if } j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Consider next the case of a positive state $i$, and suppose that a transition occurs at some time $\bar{t}$ to $X(\bar{t}) = i$. If the next transition occurs at time $\bar{t}+\Delta_i$, then $\Delta_i$ is the minimum of two exponentially distributed random variables: the time to the next arrival, call it Y, which has parameter $\lambda$, and the time to the next departure, call it $Z$, which has parameter $\mu$. (We are again using here the memoryless property of the exponential distribution.) Thus according to Example 5.15, which deals with "competing exponentials," the time $\Delta_i$ is exponentially distributed with parameter $\nu_i = \lambda + \mu$. Furthermore, the probability that the next transition corresponds to an arrival is

$$\begin{aligned}
\mathbf{P}(Y \le Z) &= \int_{y \le z} \lambda e^{-\lambda y} \cdot \mu e^{\mu z}\, dy\, dz \\
&= \lambda\mu \int_0^\infty e^{-\lambda y} \left( \int_y^\infty e^{\mu z}\, dz \right) dy \\
&= \lambda\mu \int_0^\infty e^{-\lambda y} \left( \frac{e^{-\mu y}}{\mu} \right) dy \\
&= \lambda \int_0^\infty e^{-(\lambda+\mu) y}\, dy \\
&= \frac{\lambda}{\lambda + \mu}.
\end{aligned}$$

We thus have for $i > 0$, $q_{i,i+1} = \nu_i \mathbf{P}(Y \le Z) = (\lambda + \mu)\big(\lambda/(\lambda+\mu)\big) = \lambda$. Similarly, we obtain that the probability that the next transition corresponds to a departure is $\mu/(\lambda+\mu)$, and we have $q_{i,i-1} = \nu_i \mathbf{P}(Y \ge Z) = (\lambda+\mu)\big(\mu/(\lambda+\mu)\big) = \mu$. Thus

$$q_{ij} = \begin{cases} \lambda & \text{if } j = i+1, \\ \mu & \text{if } j = i-1, \\ 0 & \text{otherwise.} \end{cases}$$

The positive transition rates $q_{ij}$ are recorded next to the arcs $(i, j)$ of the transition diagram, as in Fig. 6.22.

We will be interested in chains for which the discrete-time Markov chain corresponding to the transition probabilities $p_{ij}$ satisfies the accessibility and

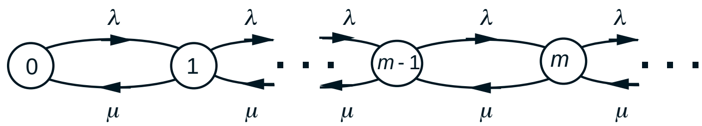



**Figure 6.22:** Transition graph for the $M/M/1$ queue (Example 6.17).

aperiodicity assumptions of the preceding section. We also require a technical condition, namely that the number of transitions in any finite length of time is finite with probability one. Almost all models of practical use satisfy this condition, although it is possible to construct examples that do not.

Under the preceding conditions, it can be shown that the limit

$$\pi_j = \lim_{t\to\infty} P\big(X(t) = j \,|\, X(0) = i\big)$$

exists and is independent of the initial state $i$. We refer to $\pi_j$ as the steady-state probability of state $j$. It can be shown that if $T_j(t)$ is the expected value of the time spent in state $j$ up to time $t$, then, regardless of the initial state, we have

$$\pi_j = \lim_{t\to\infty} \frac{T_j(t)}{t}$$

that is, $\pi_j$ can be viewed as the long-term proportion of time the process spends in state $j$.

The balance equations for a continuous-time Markov chain take the form

$$p_j \sum_{i=0}^{\infty} q_{ji} = \sum_{i=0}^{\infty} p_i q_{ij}, \qquad j = 0, 1, \ldots$$

Similar to discrete-time Markov chains, it can be shown that there are two possibilities:

(1) The steady-state probabilities are all positive and solve uniquely the balance equations together with the normalization equation $\pi_1 + \pi_2 + \cdots = 1$.

(2) The steady-state probabilities are all zero.

To interpret the balance equations, we note that since $\pi_i$ is the proportion of time the process spends in state $i$, it follows that $\pi_i q_{ij}$ can be viewed as frequency of transitions from $i$ to $j$ (expected number of transitions from $i$ to $j$ per unit time). It is seen therefore that the balance equations express the intuitive fact that the frequency of transitions out of state $j$ (the left side term $\pi_j \sum_{i=1}^{\infty} q_{ji}$) is equal to the frequency of transitions into state $j$ (the right side term $\sum_{i=0}^{\infty} \pi_i q_{ij}$).

The continuous-time analog of the local balance equations for discrete-time chains is

$$\pi_j q_{ji} = \pi_i q_{ij}, \qquad i, j = 1, 2, \ldots$$

These equations hold in birth-death systems where $q_{ij} = 0$ for $|i - j| > 1$, but need not hold in other types of Markov chains. They express the fact that the frequencies of transitions from $i$ to $j$ and from $j$ to $i$ are equal.

To understand the relationship between the balance equations for continuous-time chains and the balance equations for discrete-time chains, consider any $\delta > 0$, and the discrete-time Markov chain $\{Z_n \mid n \geq 0\}$, where

$$Z_n = X(n\delta), \qquad n = 0, 1, \ldots$$

The steady-state distribution of $\{Z_n\}$ is clearly $\{\pi_j \mid j \geq 0\}$, the steady-state distribution of the continuous chain. The transition probabilities of $\{Z_n \mid n \geq 0\}$ can be derived by using the properties of the exponential distribution. We obtain

$$\overline{p}_{ij} = \delta q_{ij} + o(\delta), \qquad i \neq j,$$

$$\overline{p}_{jj} = 1 - \delta \sum_{\substack{i=0 \\ i \neq j}}^{\infty} q_{ji} + o(\delta)$$

Using these expressions, the balance equations

$$\pi_j = \sum_{i=0}^{\infty} \pi_i \, \overline{p}_{ij} \qquad j \geq 0$$

for the discrete-time chain $\{Z_n\}$, we obtain

$$\pi_j = \sum_{i=0}^{\infty} \pi_i p_{ij} = p_j \big(1 - \delta \sum_{\substack{i=0 \\ i \neq j}}^{\infty} q_{ji} + o(\delta)\big) + \sum_{\substack{i=0 \\ i \neq j}} p_i \big(\delta q_{ij} + o(\delta)\big).$$

Taking the limit as $\delta \to 0$, we obtain the balance equations for the continuous-time chain.

**Example 6.18. (The $M/M/1$ Queue – Continued)** As in the case of a finite number of states, the local balance equations are

$$\pi_i \lambda = \pi_{i+1} \mu, \qquad i = 0, 1, \ldots,$$

and we obtain $\pi_{i+1} = \rho \pi_i$, where $\rho = \lambda/\mu$. Thus, we have $\pi_i = \rho^i \pi_0$ for all $i$. If $\rho < 1$, the normalization equation $1 = \sum_{i=0}^{\infty} \pi_i$ yields

$$1 = \pi_0 \sum_{i=0}^{\infty} \rho^i = \frac{\pi_0}{1 - \rho},$$

in which case $\pi_0 = 1 - \rho$, and the steady-state probabilities are

$$\pi_i = \rho^i (1 - \rho), \qquad i = 0, 1, \ldots$$

If $\rho \geq 1$, which corresponds to the case where the arrival probability $b$ is no less than the departure probability $d$, the normalization equation $1 = \pi_0(1+\rho+\rho^2+\cdots)$ implies that $\pi_0 = 0$, and also $\pi_i = \rho^i \pi_0 = 0$ for all $i$.

**Example 6.19. (The $M/M/m$ and $M/M/\infty$ Queues)** The $M/M/m$ queueing system is identical to the $M/M/1$ system except that $m$ packets can be simultaneously transmitted (i.e., the transmission line of the node has $m$ transmission channels). A packet at the head of the queue is routed to any channel that is available. The corresponding state transition diagram is shown in Fig. 6.24.

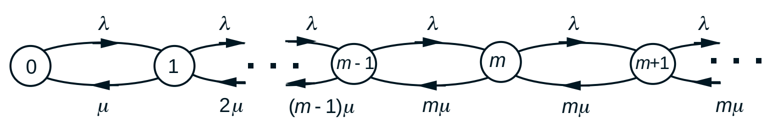



**Figure 6.24:** Transition graph for the $M/M/m$ queue (Example 6.19).

By writing down the local balance equations for the steady-state probabilities $\pi_n$, we obtain

$$\lambda \pi_{n-1} = \begin{cases} n\mu\pi_n & \text{if } n \leq m, \\ m\mu\pi_n & \text{if } n > m. \end{cases}$$

From these equations, we obtain

$$\pi_n = \begin{cases} p_0 \dfrac{(m\rho)^n}{n!}, & n \leq m \\[2em] p_0 \dfrac{m^m \rho^n}{m!}, & n > m \end{cases}$$

where $\rho$ is given by

$$\rho = \frac{\lambda}{m\mu}.$$

Assuming $\rho < 1$, we can calculate $\pi_0$ using the above equations and the condition $\sum_{n=0}^{\infty} \pi_n = 1$. We obtain

$$\pi_0 = \left( 1 + \sum_{n=1}^{m-1} \frac{(m\rho)^n}{n!} + \sum_{n=m}^{\infty} \frac{(m\rho)^n}{m!} \frac{1}{m^{n-m}} \right)^{-1}$$

and, finally,

$$\pi_0 = \left( \sum_{n=0}^{m-1} \frac{(m\rho)^n}{n!} + \frac{(m\rho)^m}{m!(1-\rho)} \right)^{-1}.$$

In the limiting case where $m = \infty$ in the $M/M/m$ system (which is called the $M/M/\infty$ system), the local balance equations become

$$\lambda \pi_{n-1} = n\mu\pi_n, \qquad n = 1, 2, \ldots$$

so

$$\pi_n = \pi_0 \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!}, \qquad n = 1, 2, \ldots$$

From the condition $\sum_{n=0}^{\infty} \pi_n = 1$, we obtain

$$\pi_0 = \left(1 + \sum_{n=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!}\right)^{-1} = e^{-\lambda/\mu},$$

so, finally,

$$\pi_n = \left(\frac{\lambda}{\mu}\right)^n \frac{e^{-\lambda/\mu}}{n!}, \qquad n = 0, 1, \ldots$$

Therefore, in steady-state, the number in the system is Poisson distributed with parameter $\lambda/\mu$.