2

MAXWELL'S DIFFERENTIAL LAWS IN FREE SPACE

2.0 INTRODUCTION

Maxwell's integral laws encompass the laws of electrical circuits. The transition from fields to circuits is made by associating the relevant volumes, surfaces, and contours with electrodes, wires, and terminal pairs. Begun in an informal way in Chap. 1, this use of the integral laws will be formalized and examined as the following chapters unfold. Indeed, many of the empirical origins of the integral laws are in experiments involving electrodes, wires and the like.

The remarkable fact is that the integral laws apply to any combination of volume and enclosing surface or surface and enclosing contour, whether associated with a circuit or not. This was implicit in our use of the integral laws for deducing field distributions in Chap. l.

Even though the integral laws can be used to determine the fields in highly symmetric configurations, they are not generally applicable to the analysis of realistic problems. Reasons for this lie beyond the geometric complexity of practical systems. Source distributions are not generally known, even when materials are idealized as insulators and "perfect" conductors. In actual materials, for example, those having finite conductivity, the self-consistent interplay of fields and sources, must be described.

Because they apply to *arbitrary* volumes, surfaces, and contours, the integral laws also contain the differential laws that apply at each point in space. The differential laws derived in this chapter provide a more broadly applicable basis for predicting fields. As might be expected, the point relations must involve information about the shape of the fields in the neighborhood of the point. Thus it is that the integral laws are converted to point relations by introducing partial derivatives of the fields with respect to the spatial coordinates.

The plan in this chapter is first to write each of the integral laws in terms of one type of integral. For example, in the case of Gauss' law, the surface integral is converted to one over the volume V enclosed by the surface.

$$\int_V div(\epsilon_o \mathbf{E}) dv = \oint_S \epsilon_o \mathbf{E} \cdot d\mathbf{a} \tag{1}$$

Here div is some combination of spatial derivatives of $\epsilon_o \mathbf{E}$ to be determined in the next section. With this mathematical theorem accepted for now, Gauss' integral law, (1.3.1), can be written in terms of volume integrals.

$$\int_V div(\epsilon_o \mathbf{E}) dv = \int_V \rho dv \tag{2}$$

The desired differential form of Gauss' law is obtained by equating the integrands in this expression.

$$div(\epsilon_o \mathbf{E}) = \rho \tag{3}$$

Is it true that if two integrals are equal, their integrands are as well? In general, the answer is no! For example, if x^2 is integrated from 0 to 1, the result is the same as for an integration of $2x/3$ over the same interval. However, x^2 is hardly equal to $2x/3$ for every value of x.

It is because the volume V is arbitrary that we can equate the integrands in (1). For a one-dimensional integral, this is equivalent to having endpoints that are arbitrary. With the volume arbitrary (the endpoints arbitrary), the integrals can only be equal if the integrands are as well.

The equality of the three-dimensional volume integration on the left in (1) and the two-dimensional surface integration on the right is analogous to the case of a one-dimensional integral being equal to the function evaluated at the integration endpoints. That is, suppose that the operator der operates on $f(x)$ in such a way that

$$\int_{x_1}^{x_2} der(f) dx = f(x_2) - f(x_1) \tag{4}$$

The integration on the left over the "volume" interval between x_1 and x_2 is reduced by this "theorem" to an evaluation on the "surface," where $x = x_1$ and $x = x_2$.

The procedure for determining the operator der in (4) is analogous to that used to deduce the divergence and curl operators in Secs. 2.1 and 2.4, respectively. The point x at which der is to be evaluated is taken midway in the integration interval, as in Fig. 2.0.1. Then the interval is taken as incremental ($\Delta x = x_2 - x_1$) and for small Δx, (4) becomes

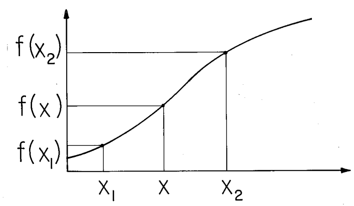

Fig. 2.0.1 General function of x defined between endpoints x_1 and x_2.

$$[der(f)]\Delta x = f(x_2) - f(x_1) \tag{5}$$

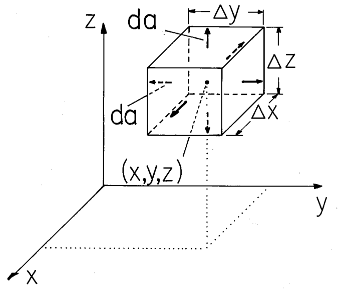

Fig. 2.1.1 Incremental volume element for determination of divergence operator.

It follows that

$$der = \lim_{\Delta x \to 0} \left[\frac{f\left(x + \frac{\Delta x}{2}\right) - f\left(x - \frac{\Delta x}{2}\right)}{\Delta x} \right] \tag{6}$$

Thus, as we knew to begin with, *der* is the derivative of f with respect to x.

Byproducts of the derivation of the divergence and curl operators in Secs. 2.1 and 2.4 are the integral theorems of Gauss and Stokes, derived in Secs. 2.2 and 2.5, respectively. A *theorem* is a mathematical relation and must be distinguished from a physical *law*, which establishes a physical relation among physical variables. The differential laws, together with the operators and theorems that are the point of this chapter, are summarized in Sec. 2.8.

2.1 THE DIVERGENCE OPERATOR

If Gauss' integral theorem, (1.3.1), is to be written with the surface integral replaced by a volume integral, then it is necessary that an operator be found such that

$$\int_V div \mathbf{A} dv = \oint_S \mathbf{A} \cdot d\mathbf{a} \tag{1}$$

With the objective of finding this *divergence operator*, *div*, (1) is applied to an incremental volume ΔV. Because the volume is small, the volume integral on the left can be taken as the product of the integrand and the volume. Thus, the divergence of a vector $\mathbf{A}$ is defined in terms of the limit of a surface integral.

$$div \mathbf{A} \equiv \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint_S \mathbf{A} \cdot d\mathbf{a} \tag{2}$$

Once evaluated, it is a function of $\mathbf{r}$. That is, in the limit, the volume shrinks to zero in such a way that all points on the surface approach the point $\mathbf{r}$. With this condition satisfied, the actual shape of the volume element is arbitrary.

In Cartesian coordinates, a convenient incremental volume is a rectangular parallelepiped $\Delta x \Delta y \Delta z$ centered at (x, y, z), as shown in Fig. 2.1.1. With the limit where $\Delta x \Delta y \Delta z \to 0$ in view, the right-hand side of (2) is approximated by

$$\begin{aligned}\oint_S \mathbf{A}\cdot d\mathbf{a} \simeq \Delta y\Delta z\big[A_x\big(x+\frac{\Delta x}{2},y,z\big)-A_x\big(x-\frac{\Delta x}{2},y,z\big)\big]\\ +\Delta z\Delta x\big[A_y\big(x,y+\frac{\Delta y}{2},z\big)-A_y\big(x,y-\frac{\Delta y}{2},z\big)\big]\\ +\Delta x\Delta y\big[A_z\big(x,y,z+\frac{\Delta z}{2}\big)-A_z\big(x,y,z-\frac{\Delta z}{2}\big)\big]\end{aligned} \tag{3}$$

With the above expression used to evaluate (2), along with $\Delta V = \Delta x\Delta y\Delta z$,

$$\begin{aligned} div\mathbf{A} = \lim_{\Delta x\to 0}\left[\frac{A_x\big(x+\frac{\Delta x}{2},y,z\big)-A_x\big(x-\frac{\Delta x}{2},y,z\big)}{\Delta x}\right]\\ +\lim_{\Delta y\to 0}\left[\frac{A_y\big(x,y+\frac{\Delta y}{2},z\big)-A_y\big(x,y-\frac{\Delta y}{2},z\big)}{\Delta y}\right]\\ +\lim_{\Delta z\to 0}\left[\frac{A_z\big(x,y,z+\frac{\Delta z}{2}\big)-A_z\big(x,y,z-\frac{\Delta z}{2}\big)}{\Delta z}\right]\end{aligned} \tag{4}$$

It follows that in Cartesian coordinates, the divergence operator is

$$\boxed{div\mathbf{A} = \frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}} \tag{5}$$

This result suggests an alternative notation. The *del* operator is defined as

$$\boxed{\nabla \equiv \mathbf{i_x}\frac{\partial}{\partial x}+\mathbf{i_y}\frac{\partial}{\partial y}+\mathbf{i_z}\frac{\partial}{\partial z}} \tag{6}$$

so that (5) can be written as

$$div\mathbf{A} = \nabla\cdot\mathbf{A} \tag{7}$$

The *div* notation suggests that this combination of derivatives describes the outflow of $\mathbf{A}$ from the neighborhood of the point of evaluation. The definition (2) is independent of the choice of a coordinate system. On the other hand, the *del* notation suggests the mechanics of the operation in Cartesian coordinates. We will have it both ways by using the *del* notation in writing equations in Cartesian coordinates, but using the name divergence in the text.

Problems 2.1.4 and 2.1.6 lead to the divergence operator in cylindrical and spherical coordinates, respectively (summarized in Table I at the end of the text), and provide the opportunity to develop the connection between the general definition, (2), and specific representations.

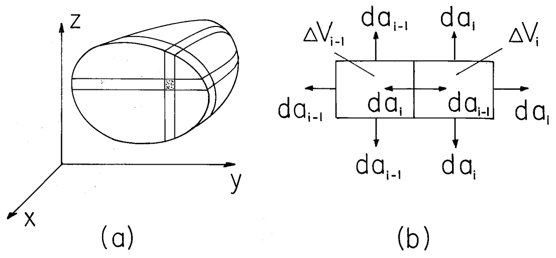

Fig. 2.2.1 (a) Three mutually perpendicular slices define an incremental volume in the volume V shown in cross-section. (b) Adjacent volume elements with common surface.

2.2 GAUSS' INTEGRAL THEOREM

The operator that is required for (2.1.1) to hold has been identified by considering an incremental volume element. But does the relation hold for volumes of finite size?

The volume enclosed by the surface S can be subdivided into differential elements, as shown in Fig. 2.2.1. Each of the elements has a surface of its own with the i-th being enclosed by the surface S_i. We now prove that the surface integral of the vector $\mathbf{A}$ over the surface S is equal to the sum of the surface integrals over each surface S

$$\oint_S \mathbf{A} \cdot d\mathbf{a} = \sum_i \Big[\int_{S_i} \mathbf{A} \cdot d\mathbf{a} \Big] \tag{1}$$

Note first that the surface normals of two surfaces between adjacent volume elements are oppositely directed, while the vector $\mathbf{A}$ has the same value for both surfaces. Thus, as illustrated in Fig. 2.2.1, the fluxes through surfaces separating two volume elements in the interior of S cancel.

The only contributions to the summation in (1) which do not cancel are the fluxes through the surfaces which do not separate one volume element from another, i.e., those surfaces that lie on S. But because these surfaces together form S, (1) follows. Finally, with the right-hand side rewritten, (1) is

$$\oint_S \mathbf{A} \cdot d\mathbf{a} = \sum_i \Big[\frac{\int_{S_i} \mathbf{A} \cdot d\mathbf{a}}{\Delta V_i} \Big] \Delta V_i \tag{2}$$

where ΔV_i is the volume of the i-th element. Because these volume elements are differential, what is in brackets on the right in (2) can be represented using the definition of the divergence operator, (2.1.2).

$$\oint_S \mathbf{A} \cdot d\mathbf{a} = \sum_i (\nabla \cdot \mathbf{A})_i \Delta V_i \tag{3}$$

Gauss' integral theorem follows by replacing the summation over the differential volume elements by an integration over the volume.

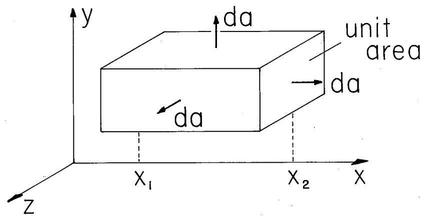

Fig. 2.2.2 Volume between planes $x = x_1$ and $x = x_2$ having unit area in $y - z$ planes.

$$\boxed{\oint_S \mathbf{A} \cdot d\mathbf{a} = \int_V \nabla \cdot \mathbf{A} dv} \tag{4}$$

Example 2.2.1. One-Dimensional Theorem

If the vector $\mathbf{A}$ is one-dimensional so that

$$\mathbf{A} = f(x)\mathbf{i_x} \tag{5}$$

what does Gauss' integral theorem say about an integration over a volume V between the planes $x = x_1$ and $x = x_2$ and of unit cross-section in any $y - z$ plane between these planes? The volume V and surface S are as shown in Fig. 2.2.2. Because $\mathbf{A}$ is x directed, the only contributions are from the right and left surfaces. These respectively have $d\mathbf{a} = \mathbf{i_x} dydz$ and $d\mathbf{a} = -\mathbf{i_x} dydz$. Hence, substitution into (4) gives the familiar form,

$$f(x_2) - f(x_1) = \int_{x_1}^{x_2} \frac{\partial f}{\partial x} dx \tag{6}$$

which is a reminder of the one-dimensional analogy discussed in the introduction. Gauss' theorem extends into three dimensions the relationship that exists between the derivative and integral of a function.

2.3 GAUSS' LAW, MAGNETIC FLUX CONTINUITY, AND CHARGE CONSERVATION

Of the five integral laws summarized in Table 1.8.1, three involve integrations over closed surfaces. By Gauss' theorem, (2.2.4), each of the surface integrals is now expressed as a volume integral. Because the volume is arbitrary, the integrands must vanish, and so the differential laws are obtained.

The *differential form of Gauss' law* follows from (1.3.1) in that table.

$$\boxed{\nabla \cdot \epsilon_o \mathbf{E} = \rho} \tag{1}$$

Magnetic flux continuity in differential form follows from (1.7.1).

$$\boxed{\nabla \cdot \mu_o \mathbf{H} = 0} \tag{2}$$

In the integral charge conservation law, (1.5.2), there is a time derivative. Because the geometry of the integral we are considering is fixed, the time derivative can be taken inside the integral. That is, the spatial integration can be carried out after the time derivative has been taken. But because ρ is not only a function of t but of (x, y, z) as well, the time derivative is taken holding (x, y, z) constant. Thus, the *differential charge conservation law* is stated using a partial time derivative.

$$\boxed{\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0} \tag{3}$$

These three differential laws are summarized in Table 2.8.1.

2.4 THE CURL OPERATOR

If the integral laws of Ampère and Faraday, (1.4.1) and (1.6.1), are to be written in terms of one type of integral, it is necessary to have an operator such that the contour integrals are converted to surface integrals. This operator is called the *curl.*

$$\int_S curl\ \mathbf{A} \cdot d\mathbf{a} = \oint_C \mathbf{A} \cdot d\mathbf{s} \tag{1}$$

The operator is identified by making the surface an incremental one, Δa. At the particular point $\mathbf{r}$ where the operator is to be evaluated, pick a direction $\mathbf{n}$ and construct a plane normal to $\mathbf{n}$ through the point $\mathbf{r}$. In this plane, choose a contour C around $\mathbf{r}$ that encloses the incremental area Δa. It follows from (1) that

$$(curl\ \mathbf{A})_n = \lim_{\Delta a \to 0} \frac{1}{\Delta a} \oint_C \mathbf{A} \cdot d\mathbf{s} \tag{2}$$

The shape of the contour C is arbitrary except that all its points are assumed to approach the point $\mathbf{r}$ under study in the limit $\Delta a \to 0$. Such an arbitrary elemental surface with its unit normal $\mathbf{n}$ is illustrated in Fig. 2.4.1a. The definition of the curl operator given by (2) is independent of the coordinate system.

To express (2) in Cartesian coordinates, consider the incremental surface shown in Fig. 2.4.1b. The center of Δa is at the location (x, y, z), where the operator is to be evaluated. The contour is composed of straight segments at $y \pm \Delta y/2$ and $z \pm \Delta z/2$. To first order in Δy and Δz, it follows that the $\mathbf{n} = \mathbf{i_x}$ component of (2) is

$$\begin{aligned}(curl\ \mathbf{A})_x = \lim_{\Delta y \Delta z \to 0} \frac{1}{\Delta y \Delta z} \bigg\{ &\Big[A_z\big(x, y + \frac{\Delta y}{2}, z\big) - A_z\big(x, y - \frac{\Delta y}{2}, z\big) \Big] \Delta z \\ &- \Big[A_y\big(x, y, z + \frac{\Delta z}{2}\big) - A_y\big(x, y, z - \frac{\Delta z}{2}\big) \Big] \Delta y \bigg\}\end{aligned} \tag{3}$$

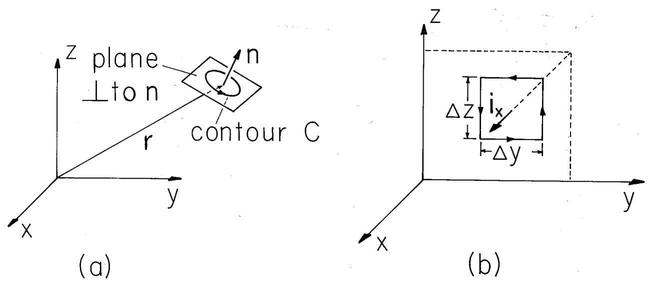

Fig. 2.4.1 (a) Incremental contour for evaluation of the component of the curl in the direction of **n**. (b) Incremental contour for evaluation of x component of curl in Cartesian coordinates.

Here the first two terms represent integrations along the vertical segments, first in the $+z$ direction and then in the $-z$ direction. Note that integration on this second leg results in a minus sign, because there, **A** is oppositely directed to $d\mathbf{s}$.

In the limit, (3) becomes

$$(curl\ \mathbf{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \tag{4}$$

The same procedure, applied to elemental areas having normals in the y and z directions, result in three "components" for the *curl* operator.

$$\begin{aligned} curl\ \mathbf{A} =& \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{i_x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{i_y} \\ &+ \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{i_z} \end{aligned} \tag{5}$$

In fact, we should be able to select the surface for evaluating (2) as having a unit normal **n** in any arbitrary direction. For (5) to be a vector, its dot product with **n** must give the same result as obtained for the direct evaluation of (2). This is shown to be true in Appendix 2.

The result of cross-multiplying **A** by the *del* operator, defined by (2.1.6), is the curl operator. This is the reason for the alternate notation for the curl operator.

$$curl\ \mathbf{A} = \nabla \times \mathbf{A} \tag{6}$$

Thus, in Cartesian coordinates

$$\boxed{\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i_x} & \mathbf{i_y} & \mathbf{i_z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}} \tag{7}$$

The problems give the opportunity to derive expressions having similar forms in cylindrical and spherical coordinates. The results are summarized in Table I at the end of the text.

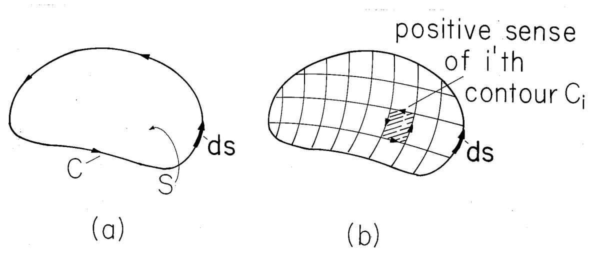

Fig. 2.5.1 Arbitrary surface enclosed by contour C is subdivided into incremental elements, each enclosed by a contour having the same sense as C.

2.5 STOKES' INTEGRAL THEOREM

In Sec. 2.4, $curl\mathbf{A}$ was identified as that vector function which had an integral over a surface S that could be reduced to an integral on $\mathbf{A}$ over the enclosing contour C. This was done by applying (2.4.1) to an incremental surface. But does this relation hold for S and C of finite size and arbitrary shape?

The generalization to an arbitrary surface begins by subdividing S into differential area elements, each enclosed by a contour C . As shown in Fig. 2.5.1, each differential contour coincides in direction with the positive sense of the original contour. We shall now prove that

$$\oint_C \mathbf{A} \cdot d\mathbf{s} = \sum_i \oint_{C_i} \mathbf{A} \cdot d\mathbf{s} \tag{1}$$

where the sum is over all contours bounding the surface elements into which the surface S has been subdivided.

Because the segments are followed in opposite senses when evaluated for the adjacent area elements, line integrals along those segments of the contours which separate two adjacent surface elements add to zero in the sum of (1). Only those line integrals remain which pertain to the segments coinciding with the original contour. Hence, (1) is demonstrated.

Next, (1) is written in the slightly different form.

$$\oint_C \mathbf{A} \cdot d\mathbf{s} = \sum_i \left[\frac{1}{\Delta a_i} \oint_{C_i} \mathbf{A} \cdot d\mathbf{s}\right] \Delta a_i \tag{2}$$

We can now appeal to the definition of the component of the curl in the direction of the normal to the surface element, (2.4.2), and replace the summation by an integration.

$$\oint_C \mathbf{A} \cdot d\mathbf{s} = \int_S (curl\ \mathbf{A})_n da \tag{3}$$

Another way of writing this expression is to take advantage of the vector character of the *curl* and the definition of a vector area element, $d\mathbf{a} = \mathbf{n}da$:

$$\boxed{\oint_C \mathbf{A} \cdot d\mathbf{s} = \int_S \nabla \times \mathbf{A} \cdot d\mathbf{a}} \tag{4}$$

This is *Stokes' integral theorem.* If a vector function can be written as the *curl* of a vector $\mathbf{A}$, then the integral of that function over a surface S can be reduced to an integral of $\mathbf{A}$ on the enclosing contour C.

2.6 DIFFERENTIAL LAWS OF AMPÈRE AND FARADAY

With the help of Stokes' theorem, Ampère's integral law (1.4.1) can now be stated as

$$\int_S \nabla \times \mathbf{H} \cdot d\mathbf{a} = \int_S \mathbf{J} \cdot d\mathbf{a} + \frac{d}{dt} \int_S \epsilon_o \mathbf{E} \cdot d\mathbf{a} \tag{1}$$

That is, by virtue of (2.5.4), the contour integral in (1.4.1) is replaced by a surface integral. The surface S is fixed in time, so the time derivative in (1) can be taken inside the integral. Because S is also arbitrary, the integrands in (1) must balance.

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \epsilon_o \mathbf{E}}{\partial t}} \tag{2}$$

This is the *differential form of Ampère's law.* In the last term, which is called the displacement current density, a partial time derivative is used to make it clear that the location (x, y, z) at which the expression is evaluated is held fixed as the time derivative is taken.

In Sec. 1.5, it was seen that the integral forms of Ampère's and Gauss' laws combined to give the integral form of the charge conservation law. Thus, we should expect that the differential forms of these laws would also combine to give the differential charge conservation law. To see this, we need the identity $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ (Problem 2.4.5). Thus, the divergence of (2) gives

$$0 = \nabla \cdot \mathbf{J} + \frac{\partial}{\partial t}(\nabla \cdot \epsilon_o \mathbf{E}) \tag{3}$$

Here the time and space derivatives have been interchanged in the last term. By Gauss' differential law, (2.3.1), the time derivative is of the charge density, and so (3) becomes the differential form of charge conservation, (2.3.3). Note that we are taking a differential view of the interrelation between laws that parallels the integral developments of Sec. 1.5.

Finally, Stokes' theorem converts Faraday's integral law (1.6.1) to integrations over S only. It follows that the *differential form of Faraday's law* is

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mu_o \mathbf{H}}{\partial t}} \tag{4}$$

The differential forms of Maxwell's equations in free space are summarized in Table 2.8.1.

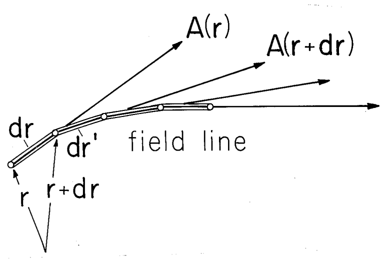

Fig. 2.7.1 Construction of field line.

2.7 VISUALIZATION OF FIELDS AND THE DIVERGENCE AND CURL

A three-dimensional vector field $\mathbf{A}(\mathbf{r})$ is specified by three components that are, individually, functions of position. It is difficult enough to plot a single scalar function in three dimensions; a plot of three is even more difficult and hence less useful for visualization purposes. Field lines are one way of picturing a field distribution.

A field line through a particular point $\mathbf{r}$ is constructed in the following way: At the point $\mathbf{r}$, the vector field has a particular direction. Proceed from the point $\mathbf{r}$ in the direction of the vector $\mathbf{A}(\mathbf{r})$ a differential distance $d\mathbf{r}$. At the new point $\mathbf{r} + d\mathbf{r}$, the vector has a new direction $\mathbf{A}(\mathbf{r} + d\mathbf{r})$. Proceed a differential distance $d\mathbf{r}'$ along this new (differentially different) direction to a new point, and so forth as shown in Fig. 2.7.1. By this process, a field line is traced out. The tangent to the field line at any one of its points gives the direction of the vector field $\mathbf{A}(\mathbf{r})$ at that point.

The magnitude of $\mathbf{A}(\mathbf{r})$ can also be indicated in a somewhat rough way by means of the field lines. The convention is used that the number of field lines drawn through an area element perpendicular to the field line at a point $\mathbf{r}$ is proportional to the magnitude of $\mathbf{A}(\mathbf{r})$ at that point. The field might be represented in three dimensions by wires.

If it has no divergence, a field is said to be *solenoidal.* If it has no curl, it is *irrotational.* It is especially important to conceptualize solenoidal and irrotational fields. We will discuss the nature of irrotational fields in the following examples, but become especially in tune with their distributions in Chap. 4. Consider now the "wire-model" picture of the solenoidal field.

Single out a surface with sides formed of a continuum of adjacent field lines, a "hose" of lines as shown in Fig. 2.7.2, with endfaces spanning across the ends of the hose. Then, because a solenoidal field can have no net flux out of this tube, the number of field lines entering the hose through one endface must be equal to the number of lines leaving the hose through the other end. Because the hose is picked arbitrarily, we conclude that a solenoidal field is represented by lines that are continuous; they do not appear or disappear within the region where they are solenoidal.

The following examples begin to develop an appreciation for the attributes of the field lines associated with the divergence and curl.

Example 2.7.1. Fields with Divergence but No Curl (Irrotational but Not Solenoidal)

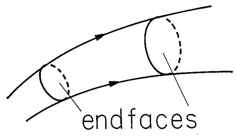

Fig. 2.7.2 Solenoidal field lines form hoses within which the lines neither begin nor end.

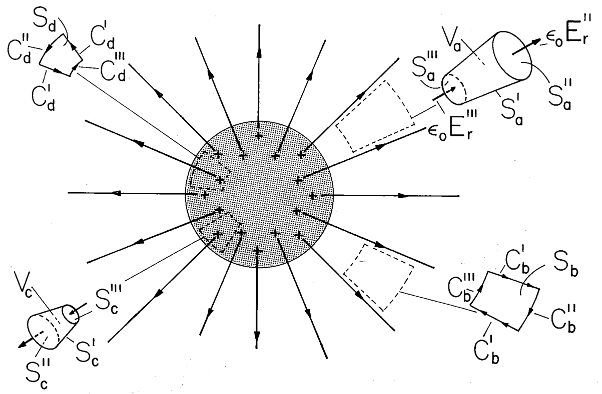

Fig. 2.7.3 Spherically symmetric field that is irrotational. Volume elements V_a and V_c are used with Gauss' theorem to show why field is solenoidal outside the sphere but has a divergence inside. Surface elements C_b and C_d are used with Stokes' theorem to show why fields are irrotational everywhere.

The spherical region $r < R$ supports a charge density $\rho = \rho_o r/R$. The exterior region is free of charge. In Example 1.3.1, the radially symmetric electric field intensity is found from the integral laws to be

$$\mathbf{E} = \mathbf{i_r} \frac{\rho_o}{4\epsilon_o} \begin{cases} \frac{r^2}{R}; & r < R \\ \frac{R^3}{r^2}; & r > R \end{cases} \tag{1}$$

In spherical coordinates, the divergence operator is (from Table I)

$$\nabla \cdot \mathbf{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta E_\theta) + \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi} \tag{2}$$

Thus, evaluation of Gauss' differential law, (2.3.1), gives

$$\epsilon_o \nabla \cdot \mathbf{E} = \begin{cases} \frac{\rho_o r}{R}; & r < R \\ 0; & r > R \end{cases} \tag{3}$$

which of course agrees with the charge distribution used in the original derivation. This exercise serves to emphasize that the differential laws apply point by point throughout the region.

The field lines can be sketched as in Fig. 2.7.3. The magnitude of the charge density is represented by the density of + (or −) symbols.

Where in this plot does the field have a divergence? Because the charge density has already been pictured, we already know the answer to this question. The field has divergence only where there is a charge density. Thus, even though the field lines are thinning out with increasing radius in the exterior region, at any given point in this region the field has no divergence. The situation in this region is typified by the flux of $\mathbf{E}$ through the "hose" defined by the volume V_a. The field does indeed decrease with radius, but the cross-sectional area of the hose increases so as to exactly compensate and maintain the net flux constant.

In the interior region, a volume element having the shape of a tube with sides parallel to the radial field can also be considered, volume V_c. That the field is not solenoidal is evident from the fact that its intensity is least over the cross-section of the tube having the least area. That there must be a net outward flux is evidence of the net charge enclosed. Field lines originate inside the volume on the enclosed charges.

Are the field lines in Fig. 2.7.3 irrotational? In spherical coordinates, the *curl* is

$$\begin{aligned}\nabla\times\mathbf{E} =&\mathbf{i_r}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(E_\phi\sin\theta)-\frac{\partial E_\theta}{\partial\phi}\right]\\&+\mathbf{i}_\theta\left[\frac{1}{r\sin\theta}\frac{\partial E_r}{\partial\phi}-\frac{1}{r}\frac{\partial}{\partial r}(rE_\phi)\right]\\&+\mathbf{i}_\phi\left[\frac{1}{r}\frac{\partial}{\partial r}(rE_\theta)-\frac{1}{r}\frac{\partial E_r}{\partial\theta}\right]\end{aligned}\tag{4}$$

and it follows from a substitution of (1) that there is no curl, either inside or outside. This result is corroborated by evaluating the circulation of $\mathbf{E}$ for contours enclosing areas Δa having normals in any one of the coordinate directions. [Remember the definition of the *curl*, (2.4.2).] Examples are the contours enclosing the surfaces S_b and S_d in Fig. 2.7.3. Contributions to the C'' and C''' segments vanish because these are perpendicular to $\mathbf{E}$, while (because $\mathbf{E}$ is independent of ϕ and θ) the contribution from one C' segment cancels that from the other.

Example 2.7.2. Fields with Curl but No Divergence (Solenoidal but Not Irrotational)

A wire having radius R carries an axial current density that increases linearly with radius. Ampère's integral law was used in Example 1.4.1 to show that the associated magnetic field intensity is

$$\mathbf{H}=\mathbf{i}_\phi\frac{J_o}{3}\begin{cases}r^2/R; & r<R\\ R^2/r; & r>R\end{cases}\tag{5}$$

Where does this field have curl? The answer follows from Ampère's law, (2.6.2), with the displacement current neglected. The curl is the current density, and hence restricted to the region $r<R$, where it tends to be concentrated at the periphery. Evaluation of the *curl* in cylindrical coordinates gives a result consistent with this expectation.

$$\begin{aligned}\nabla\times\mathbf{H}=&\mathbf{i_r}\Big(\frac{1}{r}\frac{\partial H_z}{\partial\phi}-\frac{\partial H_\phi}{\partial z}\Big)+\mathbf{i}_\phi\Big(\frac{\partial H_r}{\partial z}-\frac{\partial H_z}{\partial r}\Big)\\&+\mathbf{i_z}\Big(\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi)-\frac{1}{r}\frac{\partial H_r}{\partial\phi}\Big)\\=&\begin{cases}J_o r/R\mathbf{i_z}; & r<R\\ 0; & r>R\end{cases}\end{aligned}\tag{6}$$

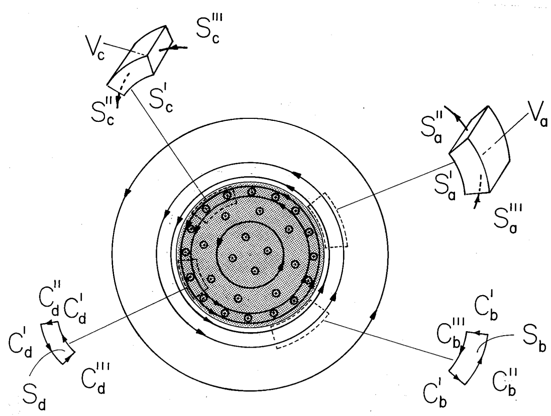

Fig. 2.7.4 Cylindrically symmetric field that is solenoidal. Volume elements V_a and V_c are used with Gauss' theorem to show why the field has no divergence anywhere. Surface elements S_b and S_d are used with Stokes' theorem to show that the field is irrotational outside the cylinder but does have a curl inside.

The current density and magnetic field intensity are sketched in Fig. 2.7.4. In accordance with the "wire" representation, the spacing of the field lines indicates their intensity. A similar convention applies to the current density. When seen "end-on," a current density headed out of the paper is indicated by $\odot$, while $\otimes$ indicates the vector is headed into the paper. The suggestion is of the vector pictured as an arrow, with the symbols representing its tip and feathers, respectively.

Can the azimuthally directed field vary with r (a direction perpendicular to ϕ) and still have no curl in the outer region? The integration of $\mathbf{H}$ around the contour C_b in Fig. 2.7.4 shows why it can. The contours C_b' are arranged to make $d\mathbf{s}$ perpendicular to $\mathbf{H}$, so that $\mathbf{H} \cdot d\mathbf{s} = 0$ there. Integrations on the segments C_b''' and C_b'' cancel because the difference in the length of the segments just compensates the decrease in the field with radius.

In the interior region, a similar integration surely gives a finite result. On the contour C_d, the field is larger on the outside leg where the contour length is larger, so it is clear that the curl must be finite. Of course, this field shape simply reflects the presence of the current density.

The field is solenoidal everywhere. This can be checked by taking the divergence of (5) in each of the regions. In cylindrical coordinates, Table I gives

$$\nabla \cdot \mathbf{H} = \frac{1}{r}\frac{\partial}{\partial r}(rH_r) + \frac{1}{r}\frac{\partial H_\phi}{\partial \phi} + \frac{\partial H_z}{\partial z} \tag{7}$$

The flux tubes defined as incremental volumes V_a and V_c in Fig. 2.7.4, in the exterior and interior regions, respectively, clearly sustain no net flux through their surfaces. That the field lines circulate in tubes without originating or disappearing in certain regions is the hallmark of the solenoidal field.

It is important to distinguish between fields "in the large" (in terms of the integral laws written for volumes, surfaces, and contours of finite size) and "in the small" (in terms of differential laws). To this end, consider some questions that might be raised.

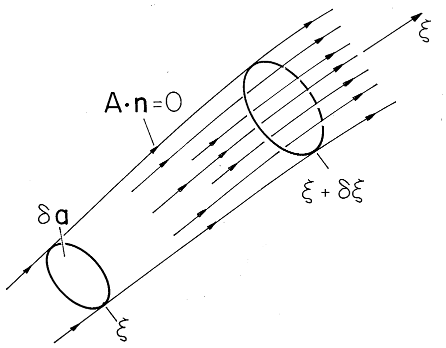

Fig. 2.7.5 Volume element with sides tangential to field lines is used to interpret divergence from field coordinate system.

Is it possible for a field that has no divergence at each point on a closed surface S to have a net flux through that surface? Example 2.7.1 illustrates that the answer is yes. At each point on a surface S that encloses the charged interior region, the divergence of $\epsilon_o\mathbf{E}$ is zero. Yet integration of $\epsilon_o\mathbf{E}\cdot d\mathbf{a}$ over such a surface gives a finite value, indeed, the net charge enclosed.

The divergence can be viewed as a weighted derivative along the direction of the field, or along the field "hose." With $\delta\mathbf{a}$ defined as the cross-sectional area of such a tube having sides parallel to the field $\epsilon_o\mathbf{E}$, as shown in Fig. 2.7.5, it follows from (2.1.2) that the divergence is

$$\nabla\cdot\mathbf{A} = \lim_{\substack{\delta a\to 0\\ \delta\xi\to 0}} \frac{1}{\delta a}\left(\frac{\mathbf{A}\cdot\delta\mathbf{a}|_{\xi+\Delta\xi} - \mathbf{A}\cdot\delta\mathbf{a}|_{\xi}}{\delta\xi}\right) \tag{8}$$

The minus sign in the second term results because $d\mathbf{a}$ and $\delta\mathbf{a}$ are negatives on the left surface. Written in this form, the divergence is the derivative of $e_o\mathbf{E}\cdot\delta\mathbf{a}$ with respect to a coordinate in the direction of $\mathbf{E}$. Examples of such tubes are volumes V_a and V_c in Fig. 2.7.3. That the divergence is zero in the exterior region of that example is equivalent to having a radial derivative of the displacement flux $\epsilon_o\mathbf{E}\cdot\delta\mathbf{a}$ that is zero.

A further observation returns to the distinction between fields as they are described "in the large" by means of the integral laws and as they are represented "in the small" by the differential laws. Is it possible for a field to have a circulation on some contour C and yet be irrotational at each point on C? Example 2.7.2 shows that the answer is again yes. The exterior magnetic field encircles the center current-carrying region. Therefore, it has a circulation on any contour that encloses the center region. Yet at all exterior points, the *curl* of $\mathbf{H}$ is zero.

The cross-product of two vectors is perpendicular to both vectors. Is the *curl* of a vector necessarily perpendicular to that vector? Example 2.7.2 would seem to say yes. There the current density is the curl of $\mathbf{H}$ and is in the z direction, while $\mathbf{H}$ is in the azimuthal direction. However, this time the answer is no. By definition we can add to $\mathbf{H}$ any irrotational field without altering the curl. If that irrotational field has a component in the direction of the curl, then the curl of the combined fields is not perpendicular to the combined fields.

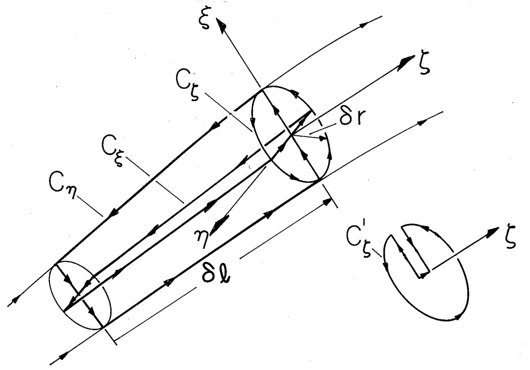

Fig. 2.7.6 Three surfaces, having orthogonal normal vectors, have geometry determined by the field hose. Thus, the curl of the field is interpreted in terms of a field coordinate system.

Illustration. A Vector Field Not Perpendicular to Its Curl

In the interior of the conductor shown in Fig. 2.7.4, the magnetic field intensity and its curl are

$$\mathbf{H} = \frac{J_o}{3}\frac{r^2}{R}\mathbf{i}_\phi; \qquad \nabla \times \mathbf{H} = \mathbf{J} = \frac{J_o r}{R}\mathbf{i_z} \tag{9}$$

Suppose that we add to this $\mathbf{H}$ a field that is uniform and z directed.

$$\mathbf{H} = \frac{J_o r^2}{3R}\mathbf{i}_\phi + H_o\mathbf{i_z} \tag{10}$$

Then the new field has a component in the z direction and yet has the same z-directed curl as given by (9). Note that the new field lines are helixes having increasingly tighter pitches as the radius is increased.

The curl can also be viewed in terms of a field hose. The definition, (2.4.2), is applied to any one of the three contours and associated surfaces shown in Fig. 2.7.6. Contours C_ξ and C_η are perpendicular and across the hose while (C_ζ) is around the hose. The former are illustrated by contours C_b and C_d in Fig. 2.7.4.

The component of the *curl* in the ξ direction is the limit in which the area $2\delta r\delta l$ goes to zero of the circulation around the contour C_ξ divided by that area. The contributions to this line integration from the segments that are perpendicular to the ζ axis are by definition zero. Thus, for this component of the *curl*, transverse to the field, (2.4.2) becomes

$$(\nabla \times \mathbf{H})_\xi = \lim_{\substack{\delta l \to 0 \\ \delta\xi \to 0}} \frac{1}{\delta l}\left(\frac{\delta \mathbf{l}\cdot\mathbf{H}\big|_{\eta+\frac{\delta\eta}{2}} - \delta\mathbf{l}\cdot\mathbf{H}\big|_{\eta-\frac{\delta\eta}{2}}}{\delta\eta}\right) \tag{11}$$

The transverse components of the *curl* can be regarded as derivatives with respect to transverse directions of the vector field weighted by incremental line elements δl.

At its center, the surface enclosed by the contour C_ζ has its normal in the direction of the field. It would seem that the curl in the ζ direction would therefore have to be zero. However, the previous discussion and illustration give a warning that the contour integral around C_ζ is not necessarily zero.

Even though, to zero order in the diameter of the hose, the field is perpendicular to the contour, to higher order it can have components parallel to the contour. This means that if the contour C_ζ were actually perpendicular to the field at each point, it would not close on itself. An equivalent contour, shown by the inset to Fig. 2.7.6, begins and terminates on the central field line. With the exception of the segment in the ζ direction used to close this contour, each segment is now by definition perpendicular to ζ. The contribution to the circulation around the contour now comes from the ζ-directed segment. Remember that the length of this segment is determined by the shape of the field lines. Thus, it is proportional to $(\delta r)^2$, and therefore so also is the circulation. The limit defined by (2.1.2) can result in a finite value in the ζ direction. The "cross-product" of an operator with a vector has properties that are not identical with the cross-product of two vectors.

2.8 SUMMARY OF MAXWELL'S DIFFERENTIAL LAWS AND INTEGRAL THEOREMS

In this chapter, the divergence and curl operators have been introduced. A third, the gradient, is naturally defined where it is put to use, in Chap. 4. A summary of these operators in the three standard coordinate systems is given in Table I at the end of the text. The problems for Secs. 2.1 and 2.4 outline the derivations of the gradient and curl operators in cylindrical and spherical coordinates.

The integral theorems of Gauss and Stokes are two of three theorems summarized in Table II at the end of the text. Gauss' theorem states how the volume integral of any scalar that can be represented as the divergence of a vector can be reduced to an integration of the normal component of that vector over the surface enclosing that volume. A volume integration is reduced to a surface integration. Similarly, Stokes' theorem reduces the surface integration of any vector that can be represented as the *curl* of another vector to a contour integration of that second vector. A surface integral is reduced to a contour integral.

These generally useful theorems are the basis for moving from the integral law point of view of Chap. 1 to a differential point of view. This transition from a global to a point-wise view of fields is summarized by the shift from the integral laws of Table 1.8.1 to the differential laws of Table 2.8.1.

The aspects of a vector field encapsulated in the divergence and curl can always be recalled by returning to the fundamental definitions, (2.1.2) and (2.4.2), respectively. The divergence is indeed defined to represent the net outward flux through a closed surface. But keep in mind that the surface is incremental, and that the divergence describes only the neighborhood of a given point. Similarly, the curl represents the circulation around an incremental contour, not around one that is of finite size.

What should be committed to memory from this chapter? The theorems of Gauss and Stokes are the key to relating the integral and differential forms of Maxwell's equations. Thus, with these theorems and the integral laws in mind,

TABLE 2.8.1

MAXWELL'S DIFFERENTIAL LAWS IN FREE SPACE

NAME	DIFFERENTIAL LAW	EQ. NUMBER
Gauss' Law	$\nabla \cdot \epsilon_o \mathbf{E} = \rho$	2.3.1
Ampère's Law	$\nabla \times \mathbf{H} = \mathbf{J} + (\partial \epsilon_o \mathbf{E})/(\partial t)$	2.6.2
Faraday's Law	$\nabla \times \mathbf{E} = -(\partial \mu_o \mathbf{H})/(\partial t)$	2.6.4
Magnetic Flux Continuity	$\nabla \cdot \mu_o \mathbf{H} = 0$	2.3.2
Charge Conservation	$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$	2.3.3

it is easy to remember the differential laws. Applied to differential volumes and surfaces, the theorems also provide the definitions (and hence the significances) of the divergence and curl operators independent of the coordinate system. Also, the evaluation in Cartesian coordinates of these operators should be remembered.

PROBLEMS

2.1 The Divergence Operator

2.1.1* In Cartesian coordinates, $\mathbf{A} = (A_o/d^2)(x^2\mathbf{i_x} + y^2\mathbf{i_y} + z^2\mathbf{i_z})$, where A_o and d are constants. Show that $div\mathbf{A} = 2A_o(x+y+z)/d^2$.

2.1.2* In Cartesian coordinates, three vector functions are

$$\mathbf{A} = (A_o/d)(y\mathbf{i_x} + x\mathbf{i_y}) \tag{a}$$

$$\mathbf{A} = (A_o/d)(x\mathbf{i_x} - y\mathbf{i_y}) \tag{b}$$

$$\mathbf{A} = A_o e^{-ky}(\cos kx\mathbf{i_x} - \sin kx\mathbf{i_y}) \tag{c}$$

where A_o, k, and d are constants.

(a) Show that the divergence of each is zero.
(b) Devise three vector functions that have a finite divergence and evaluate their divergences.

2.1.3 In cylindrical coordinates, the divergence operator is given in Table I at the end of the text. Evaluate the divergence of the following vector functions.

$$\mathbf{A} = (A_o/d)(r\cos 2\phi\mathbf{i_r} - r\sin 2\phi\mathbf{i_\phi}) \tag{a}$$

$$\mathbf{A} = A_o(\cos\phi\mathbf{i_r} - \sin\phi\mathbf{i_\phi}) \tag{b}$$

$$\mathbf{A} = (A_o r^2/d^2)\mathbf{i_r} \tag{c}$$

2.1.4* In cylindrical coordinates, unit vectors are as defined in Fig. P2.1.4a. An incremental volume element having sides $(\Delta r, r\Delta\phi, \Delta z)$ is as shown in Fig. P2.1.4b. Determine the divergence operator by evaluating (2), using steps analogous to those leading from (3) to (5). Show that the result is as given in Table I at the end of the text. (Hint: In carrying out the integrations over the surface elements in Fig. P2.1.4b having normals $\pm\mathbf{i_r}$, note that not only is A_r evaluated at $r = r \pm \frac{1}{2}\Delta r$, but so also is r. For this reason, it is most convenient to group A_r and r together in manipulating the contributions from this surface.)

2.1.5 The divergence operator is given in spherical coordinates in Table I at the end of the text. Use that operator to evaluate the divergence of the following vector functions.

$$\mathbf{A} = (A_o/d^3)r^3\mathbf{i_r} \tag{a}$$

$$\mathbf{A} = (A_o/d^2)r^2\mathbf{i_\phi} \tag{b}$$

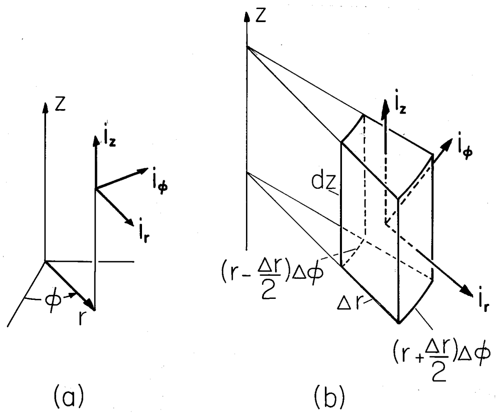

Fig. P2.1.4

$$\mathbf{A} = A_o(\cos\theta\mathbf{i_r} - \sin\theta\mathbf{i_\theta}) \tag{c}$$

2.1.6* In spherical coordinates, an incremental volume element has sides Δr, $r\Delta\theta$, $r\,\sin\theta\Delta\phi$. Using steps analogous to those leading from (3) to (5), determine the divergence operator by evaluating (2.1.2). Show that the result is as given in Table I at the end of the text.

2.2 Gauss' Integral Theorem

2.2.1* Given a well-behaved vector function $\mathbf{A}$, Gauss' theorem shows that the same result will be obtained by integrating its divergence over a volume V or by integrating its normal component over the surface S that encloses that volume. The following steps exemplify this fact. Consider the particular vector function $\mathbf{A} = (A_o/d)(x\mathbf{i_x}+y\mathbf{i_y})$ and a cubical volume having surfaces in the planes $x = \pm d$, $y = \pm d$, and $z = \pm d$.

(a) Show that the area elements on these surfaces are respectively $d\mathbf{a} = \pm\mathbf{i_x}dydz$, $\pm\mathbf{i_y}dxdz$, and $\pm\mathbf{i_z}dydx$.
(b) Show that evaluation of the left-hand side of (4) gives

$$\begin{aligned}\oint_S \mathbf{A}\cdot d\mathbf{a} = \frac{A_o}{d}\Big[&\int_{-d}^{d}\int_{-d}^{d}(d)dydz - \int_{-d}^{d}\int_{-d}^{d}(-d)dydz \\ &+\int_{-d}^{d}\int_{-d}^{d}(d)dxdz - \int_{-d}^{d}\int_{-d}^{d}(-d)dxdz\Big] \\ = 16\,&A_o d^2\end{aligned}$$

(c) Evaluate the divergence of $\mathbf{A}$ and the right-hand side of (4) and show that it gives the same result.

2.2.2 With $\mathbf{A} = (A_o/d^3)(xy^2\mathbf{i_x} + x^2y\mathbf{i_y})$, carry out the steps in Prob. 2.2.1.

2.3 Differential Forms of Gauss' Law, Magnetic Flux Continuity, and Charge Conservation

2.3.1* For a line charge along the z axis of Prob. 1.3.1, $\mathbf{E}$ was written in Cartesian coordinates as (a).

(a) Use Gauss' differential law in Cartesian coordinates to show that the charge density is indeed zero everywhere except along the z axis.
(b) Obtain the same result by evaluating Gauss' law using $\mathbf{E}$ as given by (1.3.13) and the divergence operator from Table I in cylindrical coordinates.

2.3.2* Show that at each point $r < a$, $\mathbf{E}$ and ρ as given respectively by (b) and (a) of Prob. 1.3.3 are consistent with Gauss' differential law.

2.3.3* For the flux linkage λ_f to be independent of S, (2) must hold. Return to Prob. 1.6.6 and check to see that this condition was indeed satisfied by the magnetic flux density.

2.3.4* Using $\mathbf{H}$ expressed in cylindrical coordinates by (1.4.10), show that the magnetic flux density of a line current is indeed solenoidal (has no divergence) everywhere except at $r = 0$.

2.3.5 Use the differential law of magnetic flux continuity, (2), to answer Prob. 1.7.2.

2.3.6* In Prob. 1.3.5, $\mathbf{E}$ and ρ are found for a one-dimensional configuration using the integral charge conservation law. Show that the differential form of this law is satisfied at each position $-\frac{1}{2}s < z < \frac{1}{2}s$.

2.3.7 For $\mathbf{J}$ and ρ as found in Prob. 1.5.1, show that the differential form of charge conservation, (3), is satisfied.

2.4 The Curl Operator

2.4.1* Show that the curls of the three vector functions given in Prob. 2.1.2 are zero. Devise three such functions that have finite curls (are rotational) and give their curls.

2.4.2 Vector functions are given in cylindrical coordinates in Prob. 2.1.3. Using the curl operator as given in cylindrical coordinates by Table I at the end of the text, show that all of these functions are irrotational. Devise three functions that are rotational and give their curls.

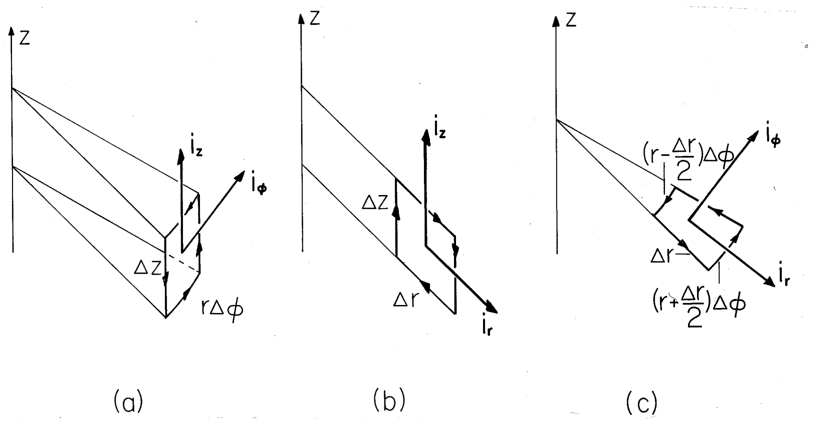

Fig. P2.4.3

2.4.3* In cylindrical coordinates, define incremental surface elements having normals in the r, ϕ and z directions, respectively, as shown in Fig. P2.4.3. Determine the r, ϕ, and z components of the curl operator. Show that the result is as given in Table I at the end of the text. (Hint: In integrating in the $\pm\phi$ directions on the outer and inner incremental contours of Fig. P2.4.3c, note that not only is A_ϕ evaluated at $r = r \pm \frac{1}{2}\Delta r$, respectively, but so also is r. It is therefore convenient to treat $A_\phi r$ as a single function.)

2.4.4 In spherical coordinates, incremental surface elements have normals in the r, θ, and ϕ directions, respectively, as described in Appendix 1. Determine the r, θ, and ϕ components of the curl operator and compare to the result given in Table I at the end of the text.

2.4.5 The following is an identity.

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \qquad (a)$$

This can be shown in two ways.

(a) Apply Stokes' theorem to an arbitrary but closed surface S (one having no edge, so $C = 0$) and then Gauss' theorem to argue the identity.
(b) Write out the the divergence of the curl in Cartesian coordinates and show that it is indeed identically zero.

2.5 Stokes' Integral Theorem

2.5.1* To exemplify Stokes' integral theorem, consider the evaluation of (4) for the vector function $\mathbf{A} = (A_o/d^2)x^2\mathbf{i_y}$ and a rectangular contour consisting of the segments at $x = g + \Delta$, $y = h$, $x = g$, and $y = 0$. The direction of the contour is such that $d\mathbf{a} = \mathbf{i_z}dxdy$.

(a) Show that the left-hand side of (4) is $hA_o[(g+\Delta)^2 - g^2]d^2$.
(b) Verify (4) by obtaining the same result integrating $curl\mathbf{A}$ over the area enclosed by C.

2.5.2 For the vector function $\mathbf{A} = (A_o/d)(-\mathbf{i_x}y + \mathbf{i_y}x)$, evaluate the contour and surface integrals of (4) on C and S as prescribed in Prob. 2.5.1 and show that they are equal.

2.6 Differential Laws of Ampère and Faraday

2.6.1* In Prob. 1.4.2, $\mathbf{H}$ is given in Cartesian coordinates by (c). With $\partial\epsilon_o\mathbf{E}/\partial t = 0$, show that Ampère's differential law is satisfied at each point $r < a$.

2.6.2* For the $\mathbf{H}$ and $\mathbf{J}$ given in Prob. 1.4.1, show that Ampère's differential law, (2), is satisfied with $\partial\epsilon_o\mathbf{E}/\partial t = 0$.

2.7 Visualization of Fields and the Divergence and Curl

2.7.1 Using the conventions exemplified in Fig. 2.7.3,

(a) Sketch the distributions of charge density ρ and electric field intensity $\mathbf{E}$ for Prob. 1.3.5 and with $E_o = 0$ and $\sigma_o = 0$.
(b) Verify that $\mathbf{E}$ is irrotational.
(c) From observation of the field sketch, why would you suspect that $\mathbf{E}$ is indeed irrotational?

2.7.2 Using Fig. 2.7.4 as a model, sketch $\mathbf{J}$ and $\mathbf{H}$

(a) For Prob. 1.4.1.
(b) For Prob. 1.4.4.
(c) Verify that in each case, $\mathbf{H}$ is solenoidal.
(d) From observation of these field sketches, why would you suspect that $\mathbf{H}$ is indeed solenoidal?

2.7.3 Three two-dimensional vector fields are shown in Fig. P2.7.3.

(a) Which of these is irrotational?
(b) Which are solenoidal?

2.7.4 For the fields of Prob. 1.6.7, sketch $\mathbf{E}$ just above and just below the plane $y = 0$ and σ_s in the surface $y = 0$. Assume that $E_1 = E_2 = \sigma_o/\epsilon_o > 0$ and adhere to the convention that the field intensity is represented by the spacing of the field lines.

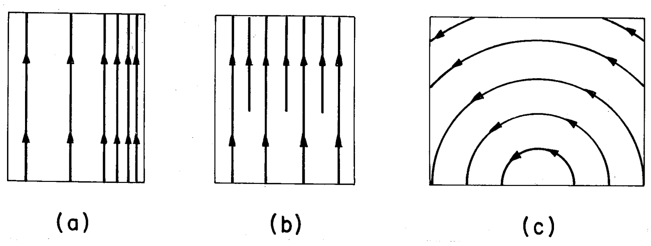

Fig. P2.7.3

2.7.5 For the fields of Prob. 1.7.3, sketch **H** just above and just below the plane $y = 0$ and **K** in the surface $y = 0$. Assume that $H_1 = H_2 = K_o > 0$ and represent the intensity of **H** by the spacing of the field lines.

2.7.6 Field lines in the vicinity of the surface $y = 0$ are shown in Fig. P2.7.6.

(a) If the field lines represent **E**, there is a surface charge density σ_s on the surface. Is σ_s positive or negative?
(b) If the field lines represent **H**, there is a surface current density $\mathbf{K} = K_z\mathbf{i_z}$ on the surface. Is K_z positive or negative?

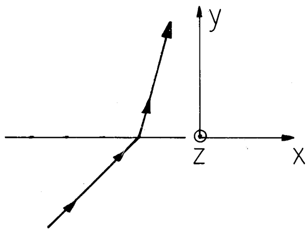

Fig. P2.7.6